Number Theory - Midterm Exam 2
December 7, 2021; 2 PM - 4.30 PM

Q 1. Let a_n denote the number of ways of putting parentheses (brackets) on a non-associative product of $n+1$ factors. For example, for $n+1=4$ factors, we have $a_3 = 5$ given by

$$((ab)c)d, (a(bc))d, a((bc)d), a(b(cd)), (ab)(cd).$$

Show that a_n is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.
Hint. Obtain a recursion such as $a_n = \frac{2(2n-1)}{n+1}a_{n-1}$.

OR

Prove that $\sum_{r=0}^{min(m,n)} \binom{m}{r}\binom{n}{r} = \binom{m+n}{m}$ in two ways: (i) counting proof, (ii) algebraic proof.

Q 2. Show that there are infinitely many natural numbers n such that $n! - 1$ has at least two different prime factors.

OR

If p_n denotes the n-th prime, prove that $p_n > 2n-1$ for $n \geq 5$ and that $p_1 p_2 \cdots p_n + 1$ is never a perfect square.

Q 3. Let k be a positive integer with at least two distinct prime factors p, q. Show that there is no positive integer n for which all the numbers $\binom{n}{i}$; $1 \leq i \leq n-1$ are multiples of k.
Hint. If $n = p^a m$ with $(p,m) = 1$, look at $\binom{n}{p^a}$.

OR

Let a, b be positive integers and let $p_1, \cdots, , p_n$ be all the primes that appear in the prime factorizations of a or b or both. If $m_1, m_2 > 1$ are co-prime positive integers, show that there exists a positive integer c such that ca is an m_1-th power and cb is an m_2-th power of a positive integer.
Hint. Writing $a = \prod_{i=1}^n p_i^{a_i}$ and $b = \prod_{i=1}^n p_i^{b_i}$ where $a_i, b_i \geq 0$, use Chinese remainder theorem to show that there exists c of the form $\prod_{i=1}^n p_i^{c_i}$ with the required property.

Q 4. Let m be a positive integer such that there exists a primitive root g mod m. Determine all the positive integers n for which the congruence $x^n \equiv g^2$ mod m has solutions for x modulo m. For such n, determine the number of solutions.

OR

Describe completely how you will obtain a solution to the congruence $x^2 \equiv -1$ mod 5^n for any given n.