Exercises

14 Dec 2021

1. Suppose A is a $m \times n$ matrix (real, cplx entries). If rank $(A) = m$ Show that $\exists$ a matrix B $(n \times m)$ s.t.

$$AB = I$$

(So that B is a right inverse of A).

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear onto. Then Show that $\exists$ a linear map $S : \mathbb{R}^m \to \mathbb{R}^n$ linear transf. s.t. $T \circ S = id$

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2. Suppose A is a $m \times n$ matrix (real, cplx) Suppose rank $(A) = n$. Then show that $\exists$ a matrix B' s.t.

$$B'A = I$$

(So B' is a left inv of A).

(Formulate a lin trans version of this exc and prove that).

③ <u>Exc + Defn</u> Suppose A is a $m \times n$ matrix (real, cplx). Suppose

$$\operatorname{rank}(A) = r \geq 1$$

Show that $\exists$ matrices P, Q s.t.

$$A = PQ$$

$P = m \times r$
$Q = r \times n$.

This pair (P, Q) is called a rank factorization of A.

Discussion on (3) <u>Thm.</u>

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Suppose

$$\begin{aligned}\operatorname{rank}(T) &= r \geq 1 \\ &\parallel \\ \dim&(T(\mathbb{R}^n)) \\ &\parallel \\ \operatorname{im}&(T)\end{aligned}$$

We claim $\exists$ linear transformations P, Q where

$$\mathbb{R}^n \xrightarrow{Q} \mathbb{R}^r \xrightarrow{P} \mathbb{R}^m$$

Such that $\underline{T = P \circ Q}$.

Generalized inv : g-inverse.

Suppose we have a system of equation

$$AX = B.$$

and assume that A is invertible, then we know that

$$A^{-1}B$$

is a solution to $AX = B$.

We might have a situation:

$$AX = B$$

is a system of equation, A is not invertible but $\exists$ a matrix G s.t. GB is a solution of $AX = B$.

Example: Consider the following

System of equation

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

A $\quad X \quad$ B.

rank$(A) = 1$, And A is a linear map

~~By rank-factorization pro~~

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

By rank factorization property $\exists$ matrices P, Q such that

$$A = PQ \qquad P = 2 \times 1$$
$$Q = 1 \times 3.$$

We can take

$$P = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

So by the first and second one, we can (take) matrices C, D s.t.

$$CP = id \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$QD = id. \qquad D = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

choice involved

Now define
$$G = DC$$

We claim: ~~$AX = B$~~ Suppose $AX = B$ has a solution say Y. In other words $AX = B$ is consistent.

Starting from A we constructed G, without any reference to B.

We now compute

$$A(GB)$$
$$= PQ(GB)$$
$$= PQDCB$$
$$= PQDCAY$$
$$= PQ\,DC\,PQ\,Y$$
$$= PQY$$
$$= AY$$
$$= B.$$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

ginv of

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

What have we achieved?

Starting from a 2×3 matrix A we constructed a 3×2 matrix G which has the property that

whenever

$$AX = B$$

is consistent, GB is a solution of

$$AX = B.$$

This motivates the definition

Defn: Let A be a $m \times n$ matrix (real, cplx). Then a matrix G $(n \times m)$ is called a generalized inv of A if GB is a solution of

$$AX = B$$

whenever $AX = B$ is consistent.

Theorem: Every $m \times n$ matrix has a generalized inverse. (Is not unique)

g-inverse.

~~Theorem: For~~

Remark: Suppose A is a $m \times n$ matrix and G is a g-inverse of A.

This means if

$$AX = B$$

is consistent, then GB is a solution of

$$AX = B.$$

If $AX = B'$ is consistent, then

GB'

is a solution of $AX = B'$.

Theorem: For any two matrices A & G, the following conditions are equivalent.

(i) G is a g-inv of A

(ii) $AGA = A$

(iii) $AGAG = AG$ & rank $(AG) =$ rank (A)

(iv) $GAGA = GA$ & rank $(GA) =$ rank (A).

Defn. A is said to be idempotent if $A^2 = A$.

$(AG)^2 = AGAG = AG$.

Proof:

(i) $\Longleftrightarrow$ (ii)

Assume that (i) holds. So assume G is a g-inv of A. We need to check

$$AGA = A.$$

Observe that we have a system of equations

$$AX = A_j$$

for each $j = 1, \ldots, n$

A_j is the j-th column of A

Is this consistent? Yes for each j this is consistent.

(i) $\Longleftarrow$ (ii)

$\Rightarrow$ GA_j is a solution of

$$AX = A_j$$

for each j.

$$\Rightarrow \quad AGA_j = A_j$$

$$\Rightarrow \quad AGA = A.$$

Thus (i) $\Rightarrow$ (ii).

Next we check (ii) $\Rightarrow$ (i).

So we assume that

$$\underline{AGA = A}$$

So we shall check $\underline{G}$ is a g-inv of A. Let $AX = B$ be consistent.

we check that

$$AGB = B.$$

y is a soln of $AX = B$.

$$\begin{aligned} AGB &= AG(Ay) \\ &= Ay \\ &= B. \end{aligned}$$

$\therefore$ G is a ginv of A.

(ii) $\Leftrightarrow$ (iii)

So assume $\underline{AGA = A}$.

$$\Rightarrow AGAG = AG.$$

Now

$$\underline{\text{rank}(AG)} \le \underline{\text{rank}(A)} = \text{rank}(AGA) \le \text{rank}(AG)$$

$$\Rightarrow \text{rank}(AG) = \text{rank}(A).$$

(Exc) check that (iii) $\Rightarrow$ (ii). (please check)

Thm: G is a g-inv of A. Then $AX = B$ is consistent $\Longleftrightarrow$ $AGB = B$.

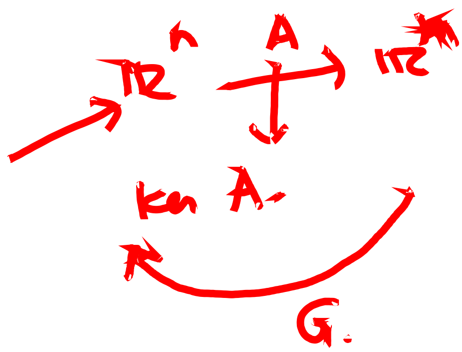
Exc + Cor: G is a g-inv of A. Then column space of C $\subseteq$ column space of A

$\Longleftrightarrow$ $AGC = C$.

Exc: Suppose A is a $m \times n$ matrix and G is a g-inverse of A. Consider the homogeneous system

$$\underline{AX = 0}$$

$A = m \times n$



Suppose $\underline{\underline{z}} \in \mathbb{R}^n$.

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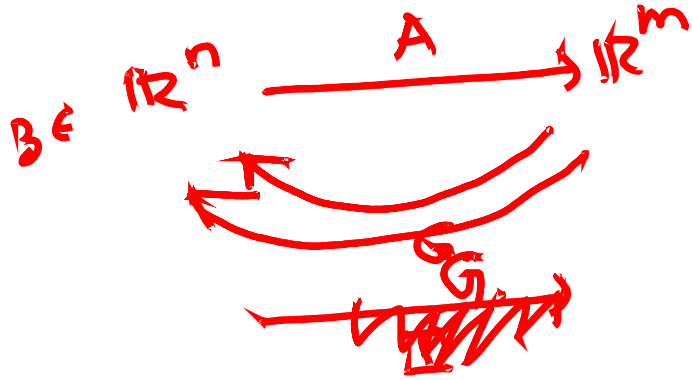
Exc: $A = m\times n$ matrix. (say real)

Suppose G is a g-inverse of A



Let $z \in \mathbb{R}^n$.

$$I,\ GA : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$(I - GA) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$A(I - GA) : \mathbb{R}^n \xrightarrow{(I-GA)} \mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$$

let $z \in \mathbb{R}^n$

$$A(I - GA)(z)$$

$$= (A - AGA)(z)$$

$$= (A - A)(z)$$

$$= 0(z) = 0.$$

$\Rightarrow (I - GA)(z)$ is a solution of

$$AX = 0.$$

Show that if y is a solution of

$$AX = 0$$

then $\exists\ z \in \mathbb{R}^n$ s.t.

$$(I - GA)z = y.$$

This characterizes all solutions of a homogen system

$$AX = 0.$$

Note that if W is the space of solution

$$W = \{ (I - GA)z \mid z \in \mathbb{R}^n \}.$$

$$\supseteq$$

$$\subseteq$$

Thm: Suppose A is a $m \times n$ matrix and $\operatorname{rank}(A) = n$.

Then a matrix G is a g-inv of A iff $GA = id$.

Suppose G is a left inv.

$\Rightarrow GA = id$.

$\Rightarrow AGA = A$

$\Rightarrow G$ is a g-inv. ✓

Conversely, suppose G is a g-inv of A.

Let C be a left inv of A.

$$AGA = A$$

$$\underbrace{CA}GA = \underbrace{CA}$$

$$GA = I$$

$\Rightarrow G$ is a left inv of A.

Exc: ~~Pro~~ Formulate & prove a similar statement when $\operatorname{rank}(A) = m$.

Cor: If A is invertible, A^{-1} is the unique g-inv ~~A^{-1}~~ of A.