16 Dec 2021

Recall

A – $m \times n$ matrix (real, cplx)
A $n \times m$ matrix G is said to be a g-inverse of A if GB is a solution of

$$AX = B$$

whenever $AX = B$ is consistent.

Thm: Every matrix has a g-inverse.

Thm: For any two matrices A, G, the foll conditions are equivalent

(i) G is a g-inv of A

(ii) $AGA = A$

(iii) $AGAG = AG$, $\operatorname{rank}(AG) = \operatorname{rank}(A)$

(iv) $GAGA = GA$, $\operatorname{rank}(GA) = \operatorname{rank}(A)$

(i) $\Leftrightarrow$ (ii)

(ii) $\Rightarrow$ (iii) — Exc (iii) $\Rightarrow$ (ii)

Assume (iii) holds. That is

$$\underline{AGAG = AG}, \quad \text{rank}(AG) = \text{rank}(A).$$

Claim: $\exists$ a matrix Q s.t.

$$GQ = I$$

$$\Rightarrow \quad \underline{AGQ = A}.$$

$$A = AGQ$$
$$= \cancel{AA}\, AG\underline{AGQ}$$
$$= AGA.$$

$$(iii) \Rightarrow (ii)$$

$\therefore$ Proof is complete.

Exc. Suppose G_1, G_2 are two g-inverses of A. Let a be a scalar. Show that

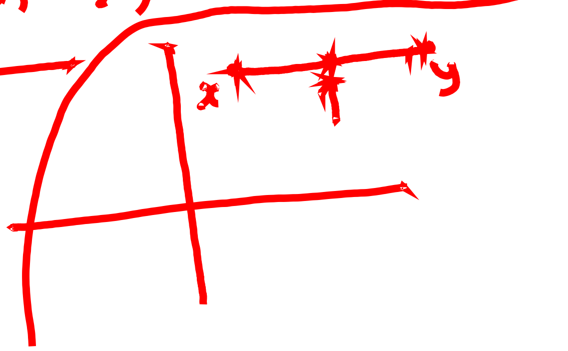
$$aG_1 + (1-a)\,G_2$$

is a g-inverse of A.

$$A\left(aG_1 + (1-a)\,G_2\right)A = A.$$

$$a\,\underbrace{AG_1A}_{A} + (1-a)\,\underbrace{AG_2A}_{A}$$

$$aA + A - aA = A.$$



- Suppose G is a g-inv of A. Then is it true that G^2 is a g-inv of A^2?

No.

non-inv matrix = singular } matrix lin mathema...

look at $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

$G = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a g-inverse.

- Suppose A is a square matrix. Let G be a g-inverse of A. Show that

$$\text{rank}(A) = tr(GA).$$

This follows from the following fact:

Suppose A is idempotent (projection) that

$$tr(A) = \text{rank}(A).$$

<u>Hint:</u> rank factorization. — rank factorize.

$$\underset{m\times n}{A} = \underset{m\times r}{B}\,\underset{r\times n}{C}$$

$r = \text{rank}(A)$.

$A^2 = A$, $BCBC = BC \Rightarrow CB = I_{r\times r}$

$\Rightarrow \operatorname{tr}(CB) = \operatorname{tr}(BC) = r$

$$\begin{aligned} &\quad\ \| \\ &\operatorname{tr}(A) = r. = \operatorname{rank}(A). \end{aligned}$$

- Suppose G is a g-inv of A. Show that $GAG = G$ if and only if $\operatorname{rank}(G) = \operatorname{rank}(A)$.

(i) $\Rightarrow$ Assume $GAG = G$.

(ii) $\Leftarrow$ assume $\operatorname{rank}(G) = \operatorname{rank}(A)$

<u>Proof of (i)</u> ✓

$$\operatorname{rank}(AG) \le \operatorname{rank}(G) \le \operatorname{rank}(GAG) \le \operatorname{rank}(AG).$$

<u>Another proof:</u>

$$\operatorname{rank}\begin{pmatrix} GAG & 0 \\ 0 & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}$$

(ii) <u>Proof of (ii)</u> (Work out the proof)

- $A = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \in \mathbb{R}^4$

What is a g-inv of A ?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & \frac{1}{3} & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

A

$A = PQ$

C

D

$G = DC$

is a g-inv.

$$\begin{pmatrix} GAG & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} G & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} AG & 0 \\ 0 & 0 \end{pmatrix}$$

Exc: Suppose
$$AX = 0$$
is a system of equations. Let W be the space of solutions of $AX = 0$. Then
Let G be a g-inv of A.
$$W = \{ (I - GA)z \mid z \in \mathbb{R}^n \} \subseteq \mathbb{R}^n$$
where A — $m \times n$ matrix with real entries.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "A"] \arrow[rr, bend right, "GA"'] \arrow[rr, bend right=60, "I"'] & \mathbb{R}^m \arrow[r, "G"] & \mathbb{R}^n. \end{tikzcd}$$

$W = \operatorname{im}(I - GA)$

We showed last time that
$$W \subseteq \operatorname{im}(I - GA). \quad \supseteq$$

Proof: (i) First show $W \subseteq \operatorname{im}(I - GA)$.

Suppose $X \in W$. $\Rightarrow AX = 0$

Then
$$(I - GA)X = IX = X$$

$\therefore X \in \operatorname{im}(I - GA) \Rightarrow W \subseteq \operatorname{im}(I - GA)$.

Proof of Lemma: $\text{im}(I_n - GA)$ is W ... Then

Suppose $X \in$ ... $= Y$. Then

Then $\exists$ a permutation matrix P s.t.

$X = (I - GA)Y$

$P^t H P = \begin{pmatrix} I_r & B \\ 0 & 0 \end{pmatrix}$.

$\Rightarrow AX = AY - AGAY$

$= 0$.

$\therefore X \in W$.

$\therefore W = \{ (I - GA)z \mid z \in \mathbb{R}^n \}$

$= \text{im}(I - GA)$.

Formulate and prove a similar observation about the set W of solution to

$AX = B$.

guess? $W = GB + \text{im}(I - GA)$.

$= \{ GB + (I - GA)z \mid z \in \mathbb{R}^n \}$