LINEAR ALGEBRA- LECTURE 1

1. Introduction

Linear algebra is the study of vector spaces and linear transformations. These terms - its definitions, examples, importance - will become clear as we study these and associated concepts in depth. Linear algebra is widely used and makes its presence felt in many braches of mathematics : multivariate calculus, probability theory, differential equations, differential geometry to name a few.

The roots of modern day Linear algebra can be traced back to the efforts of solving a system of equations. For example, let us take the following system of two linear equations

$$\begin{aligned} 4x_1 - x_2 &= 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

in two unknowns x_1 and x_2. A solution to the above system of equations is an ordered pair (x_1, x_2) (which can be thought of as a point in the plane $\mathbb{R}^2$) which satisfies both the above equations. The above system of equations has an unique solution, namely,

$$(x_1, x_2) = (1/5, 4/5).$$

This can be found out in several ways. Here are two familiar ways to find the solution.

(1) We may first solve for one of the unknowns, say x_1, in the first equation to obtain
$$x_1 = \frac{x_2}{4}$$
and substitute this in the second equation
$$\frac{x_2}{4} + x_2 = \frac{5x_2}{4} = 1$$
to get the above mentioned solution.

(2) Another familiar way to obtain a solution is by eliminating the variables. Thus, adding the two equations eliminates the variable x_2 and the solution drops out immediately.

The second method has an intrinsic advantage that will become clear soon. There is also another method, for those who are geometrically inclined, to obtain a solution to the above system of equations. One notes that the set of points that satisfy each equation is a straight line and the solution in this case is precisely the point of intersection.

Given a system of equations one is not only interested in the question of existence of solutions but also in the nature of the set of solutions if they exist. Often one could be presented with the following system of linear equations

$$\begin{aligned} 4x_1 + 4x_2 &= 0 \\ x_1 + x_2 &= 1. \end{aligned}$$

This system of linear equations clearly has no solutions. On the other hand if we have just the one system of linear equation

$$x_1 - x_2 = 0$$

then there are infinitely many solutions.

Solving a system of linear equations could become a highly complex task with the increase in the number of equations and variables. Linear algebra develops, amongst other things, a theory to undertsand the existence and nature of the solution set to a system of equations. This in a natural way then leads to the study of matrices and determinant of matrices and finally to the notion of vector spaces and linear transformations. The problem of of existence of solutions to a linear system of equations can be understood very well in the language of vector spaces and linear transformations as we shall see.

We will assume familiarity with basic notions in set theory (union, intersection, complementation) and with functions (one-one, onto). Other than this we will define everything.

We will refer to the following books : Algebra (Michael Artin), Linear Algebra (Hoffmann and Kunze) and some others given in the course details.

2. Systems of equations

We begin our study by trying to understand systems of linear equations in somewhat greater depth and generality. This will motivate our study of matrices in the next section.

A system of m linear equations in n variables is by definition a collection of m equations

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array} \tag{2.0.1}$$

in the n variables (or unknowns) $x_1, \ldots, x_n$ and where a_{ij}, b_k are numbers (either real or complex). The numbers a_{ij} are called the coefficients of the linear system and $x_1, \ldots, x_n$ the variables. The linear system (2.0.1) is said to be homogeneous if $b_1 = b_2 = \cdots = b_m = 0$ and inhomogeneous otherwise. A solution to the above linear system is defined as follows.

Definition 2.1. A solution to a system of m linear equations in n variables as in (2.0.1) is by definition a n-tuple $(s_1, \ldots, s_n)$ of numbers such that when we substitute $x_1 = s_1, \ldots, x_n = s_n$ in (2.0.1) all the equations hold.

The set of solutions to a system of linear equations is therefore a subset of the n-dimensional space (either $\mathbb{R}^n$ or $\mathbb{C}^n$). Let us try to understand our familiar method of eliminating variables in obtaining solutions to a system of linear equations by an example. Suppose we are given the system of two linear equations in three variables

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 3x_2 & + & 2x_3 & = & 0 \end{array}$$

We multiply the first equation by 3 and add it to the second equation we obtain

$$5x_1 + 5x_3 = 0$$

or $x_1 = -x_3$. So that if (x_1, x_2, x_3) is a solution, then $x_1 = -x_3$ and $x_2 = 0$. Conversely, any such triple is a solution. Thus the set of solutions consists of triples $(a, 0, -a)$. In these manipulations, it becomes increasingly clear that the manipulations are just with the coefficients and that so far as manipulations are concerned the unknowns are less important.

Let us go back to the general setup of the linear system given as in (2.0.1). Given numbers (also called scalars) $c_1, c_2, \ldots, c_m$, we multiply the i-th equation in (2.0.1) by c_i and add to get

$$(c_1 a_{11} x_1 + \cdots + c_1 a_{1n} x_n) + \cdots + (c_m a_{m1} x_1 + \cdots + c_m a_{mn} x_n) = c_1 b_1 + \cdots + c_m b_m. \tag{2.1.1}$$

We say that the equation in (2.1.1) is a linear combination of the equations in (2.0.1). It is clear that if $(x_1, \ldots, x_n)$ is a solution of (2.0.1), then it is also a solution of (2.1.1). Thus if we have another system of linear equations

$$\begin{array}{rcl} d_{11}x_1 + d_{12}x_2 + \cdots + d_{1n}x_n & = & e_1 \\ d_{21}x_1 + d_{22}x_2 + \cdots + d_{2n}x_n & = & e_2 \\ \vdots & & \vdots \\ d_{k1}x_1 + a_{k2}x_2 + \cdots + d_{kn}x_n & = & e_k \end{array} \tag{2.1.2}$$

in which each equation is a linear combination of of the equations in (2.0.1), then every solution of (2.0.1) is a solution of the system of equations in (2.1.2). Further, if every equation in (2.0.1) is also a linear combination of the equations in (2.1.2) then the two system of equations have the same set of solutions.

We say that two system of linear equations are equivalent if each equation in each system is a linear combination of the equations in the other system. Our discussion above actually proves the following.

Proposition 2.2. Equivalent systems of linear equations have exactly the same set of solutions. □

Here are two examples.

Example 2.3. Suppose we are given two systems of linear equations in two variables as below

$$\begin{array}{rcl} x_1 - x_2 & = & 0 \\ 2x_1 + x_2 & = & 0 \end{array}$$

and

$$\begin{array}{rcl} 3x_1 + x_2 & = & 0 \\ x_1 + x_2 & = & 0. \end{array}$$

We then see that

$$\frac{1}{3}(x_1 - x_2) + \frac{4}{3}(2x_1 + x_2) = 3x_1 + x_2$$

and

$$-\left(\frac{1}{3}\right)(x_1 - x_2) + \frac{2}{3}(2x_1 + x_2) = x_1 + x_2.$$

Thus each equation in the second system is a linear combination of the equations in the first system. It is an easy exercise to check that each equation in the first system is a linear combination of the equations in the second system. Thus both the linear systems have the same set of solutions.

Example 2.4. Suppose we are given a homogeneous system of m linear equations in n variables

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & 0 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & 0. \end{array} \tag{2.4.1}$$

Such a system always has a solution, namely the n-tuple $(x_1, \ldots, x_n)$ with

$$x_1 = x_2 \cdots = x_n = 0.$$

We further note the following. Suppose that $(c_1, \ldots, c_n)$ and $(d_1, \ldots, d_n)$ are two solutions of the above system. Then for each $1 \leq i \leq m$ we have

$$a_{i1}c_1 + a_{i2}c_2 + \cdots + a_{in}c_n = 0$$

and

$$a_{i1}d_1 + a_{i2}d_2 + \cdots + a_{in}d_n = 0$$

and therefore

$$a_{i1}(c_1 + d_1) + a_{i2}(c_2 + d_2) + \cdots + a_{in}(c_n + d_n) = 0.$$

Thus $(c_1 + d_1, \ldots, c_n + d_n)$ is also a solution of the above linear system. Further, for every scalar r, it is clear that $(rc_1, \ldots, rc_n)$ is also a solution of the linear system. Thus for a homogeneous system of linear equations, the set of solutions has a nice structure in that the set of solutions is closed with respect to sum and scalar mulitplication.

Here are some problems.

Exercise 2.5. Consider the two systems of linear equations in four variables

$$\begin{aligned} 2x_1 + (-1+i)x_2 + x_4 &= 0 \\ 3x_2 - 2ix_3 + 5x_4 &= 0 \end{aligned}$$

and

$$\begin{aligned} \left(1 + \tfrac{i}{2}\right) x_1 + 8x_2 - ix_3 - x_4 &= 0 \\ \left(\tfrac{2}{3}\right) x_1 - \left(\tfrac{1}{2}\right) x_2 + x_3 + 7x_4 &= 0. \end{aligned}$$

Decide whether the two systems of linear equations defined above are equivalent.

Exercise 2.6. Prove that if two homogeneous systems of linear equations in two variables have the same set of solutions, then they are equivalent.

Exercise 2.7. Which conclusions hold in Example 2.4 when the system of linear equations in consideration is inhomogeneous?

3. Matrices

Recall that one of the methods of solving a system of linear equations is that of elimination of variables. It becomes increasing evident that the manipulations that are carried out to eliminate the variables are manipulations on the coefficients of the variables and the variables themselves play a secondary role. The coefficients and the manipulations that are carried out in the method of elimination of variables can be systematically understood by use of matrices. We study these now.

Given positive integers m, n an $m \times n$ matrix is a rectangular array

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \ldots & \vdots \\ a_{mn} & a_{m2} & \ldots & \ldots & a_{mn} \end{pmatrix}$$

of mn many numbers $a_{ij} \in F$. Thus a $m \times n$ matrix has m rows and n columns and the entry a_{ij} appears in the i-th row and the j-th column. The number a_{ij} is called the ij-th entry of the matrix. The above matrix is often shortened to the notation

$$A = (a_{ij}).$$

A $m \times 1$ matrix

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

is called a m-dimensional column vector. A m-dimensional column vector thus may be identified with a point in the m-dimensional space and conversely. A $1 \times n$ matrix

$$B = \big(a_1, a_2, \cdots, a_n\big)$$

is called an n-dimensional row vector. A 1×1 matrix is a scalar. A $m \times n$ matrix is called asquare matrix if $m = n$. Two matrices are equal if and only if they are of the same size and the corresponding entries are equal.

There are various operations that one can perform on matrices. Given two $m \times n$ matrices

$$A = (a_{ij}), \quad B = (b_{ij})$$

their sum $A + B$ is defined to be the $m \times n$ matrix

$$A + B = (c_{ij})$$

where

$$c_{ij} = a_{ij} + b_{ij}.$$

The next operation is that of scalar multiplication. Given the $m \times n$ matrix A as above and a scalar r, the $m \times n$ matrix $r \cdot A$ is the matrix

$$r \cdot A = (c_{ij})$$

where

$$c_{ij} = r \cdot a_{ij}.$$

It is possible to define multiplication of matrices when the two matrices are of suitable sizes. Let

$$A = (a_{ij}), \quad B = (b_{ij})$$

be two matrices of sizes $m \times n$ and $n \times r$ respectively. The product AB is defined to be the $m \times r$ matrix

$$AB = C = (c_{ij})$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \Sigma_{i=1}^{n} a_{ik}b_{kj}.$$

In particular, the product of a row vector and a column vector

$$\big(a_1, \quad \ldots, \quad a_n\big) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1b_1 + \cdots + a_nb_n.$$

is a 1×1 matrix which is a scalar. The addition and product of matrices are connected, as one would expect, by the distributive law whenever it makes sense.

Lemma 3.1. Let A, B be two $m \times n$ matrices and let C be an $n \times r$ matrix. Then

$$(A + B)C = AC + BC.$$

Proof. We let A, B, C be the matrices

$$A = (a_{ij}), \quad B = (b_{ij}), \quad C = (c_{ij}).$$

Then $A + B = (a_{ij} + b_{ij})$ and hence if we write

$$(A + B)C = (d_{ij}),$$

then, by definition, we obtain

$$d_{ij} = \Sigma_k(a_{ik} + b_{ik})c_{kj} = \Sigma_k a_{ik}c_{kj} + \Sigma_k b_{ik}c_{kj}.$$

The last sum is precisely the sum of the ij-th terms of the matrices AC and BC respectively. This completes the proof. □

The product of matrices is associative whenever the product makes sense.

Lemma 3.2. Let A, B, C be matrices of sizes $m \times n, n \times r$ and $r \times s$ respectively and let $r \in F$ be a scalar. Then

(1) $A(BC) = (AB)C$ and
(2) $r(AB) = (rA)B = A(rB)$.

Proof. Exercise. □

Here are some exercises.

Exercise 3.3. Complete the proof of Lemma 3.2.

Exercise 3.4. Compute $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^n$.

Exercise 3.5. Find a formula for $\left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)^n$.

Exercise 3.6. Let M and M' be $m \times n$ and $n \times p$ matrices. Let r be an integer less than n. We may decompose the two matrices into blocks as follows

$$M = \begin{pmatrix} A & B \end{pmatrix}, \quad M' = \begin{pmatrix} A' \\ B' \end{pmatrix}$$

where A is a $m \times r$ matrix and A' is a $r \times p$ matrix. Show that

$$MM' = AA' + BB'.$$

We may decompose M, M' as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad M' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}.$$

Here the number of columns of A and C are equal to the number of columns of A' and C'. Verify that

$$MM' = \begin{pmatrix} AA' + BC' & AB' + BD' \\ CA' + DC' & CB' + DD' \end{pmatrix}$$

The above multiplication is called block multiplication and will be used later.

Exercise 3.7. Let A, B be square matrices. When is $(A+B)(A-B) = A^2 - B^2$. Compute $(A+B)^3$.

Exercise 3.8. In each case find all matrices that commute with the given matrix.

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Exercise 3.9. A square matrix $A = (a_{ij})$ is said to be upper triangular if $a_{ij} = 0$ for all $i > j$. Show that the product of two upper triangular matrices is an upper triangular matrix.

Exercise 3.10. A square matrix A is said to be nilpotent if $A^k = 0$ for some $k > 0$. If A is nilpotent show that $\mathbb{I} + A$ is invertible.

Exercise 3.11. Show that the product of a 2×1 matrix A and a 1×2 matrix B is not invertible.

Exercise 3.12. A square matrix $A = (a_{ij})$ is said to be symmetric if $a_{ij} = a_{ji}$ for all i, j. Check by an example that the product of two symmetric matrices need not be symmetric. Can you find a condition on two symmetric matrices A, B that will ensure that AB is symmetric?

Exercise 3.13. Find a matrix X so that

$$X\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix} = \begin{pmatrix}-7 & -8 & -9\\ 2 & 4 & 6\end{pmatrix}$$

Exercise 3.14. Suppose A and B are square matrices of the same order such that $AB = BA$. Show that $AB^n = B^nA$ for all $n > 0$.

Exercise 3.15. Let A be the matrix

$$\begin{pmatrix}1 & 0 & 1 & 4\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 1\end{pmatrix}$$

Find a column vector X so that AX is the third column of A. Find a row vector Y so that YA is the first row of A.