## LINEAR ALGEBRA- LECTURE 10

### 1. Vector spaces

Recall that a vector space over a field $F$ is a set $V$ along with two operations

$$+ : V \times V \longrightarrow V$$

called addition and another map

$$\cdot : F \times V \longrightarrow V$$

called scalar multiplication satisfying the following conditions[1]

(1) addition is commutative, that is, $u + v = v + u$ for all $u, v \in V$,
(2) addition is associative, that is, $(u + v) + ww = u + (v + w)$ for all $u, v, w \in V$,
(3) there is an additive identity, that is, there exists $0 \in V$ with $0 + u = u + u = u$ for all $u \in V$
(4) each element has an additive inverse, that is, for each $u \in V$, there exists $v \in V$ such that $u + v = 0$. The element $v$ is called the additive inverse of $u$ and we write $v = -u$.
(5) $1 \cdot v = v$ for all $v \in V$.
(6) for scalars $a, b \in F$ we have $(a + b)u = au + bu$ and $a(u + v) = au + av$ for all $u, v \in V$ and
(7) for scalars $a, b \in F$ we have $(ab)u = a(bu)$ for all $u \in V$.

If $V$ is a vector space over the field $F$, then the elements of $V$ are called vectors. Here are some properties that are easily verified.

**Lemma 1.1.** Let $V$ be a vector space over a field $F$.

(1) The additive identity in $V$ is unique.
(2) The additive inverse of a vector is unique.
(3) Let $0 \in F$ be a scalar, then $0 \cdot u = 0$ for all $u \in U$.
(4) Let $-1 \in F$ be a scalar, then $(-1)u = -u$ for all $u \in V$.

*Proof.* We first prove (1). Assume that $0, 0' \in V$ are two additive identities. Then

$$0' = 0 + 0' = 0$$

shows that $0 = 0'$. Given $u \in V$ let $u', u'' \in V$ be such that $u + u' = 0$ and $u + u'' = 0$. Then the computation

$$u' = u' + 0 = u' + (u + u'') = (u' + u) + u'' = 0 + u'' = u''$$

shows that $u' = u''$ and hence (2) holds. To prove (3) we note that

$$0 \cdot u = (0 + 0)u = 0 \cdot u + 0 \cdot u$$

and hence $0 \cdot u = 0$. To prove (4) we observe that

$$u + (-1)u = (1 + (-1))u = 0 \cdot u = 0$$

and thus $(-1)u = -u$. □

[1] In the previous set of discussion/notes we had missed out on condition (5) below. Please make note of this.

The motivation for the above abstract definition, as we have seen, comes from concrete examples. Here are some more examples.

**Example 1.2.** The addition and multiplcation of real numbers makes the set $\mathbb{R}$ of real numbers into a vector space over the field $\mathbb{R}$. Similarly, the complex numbers $\mathbb{C}$ is a vector space over itself. More generally, every field $F$ is a vector space over itself.

**Example 1.3.** Oberve that $\mathbb{C}$ and $\mathbb{R}$ are vector spaces over the field $\mathbb{Q}$ of rational numbers and that $\mathbb{C}$ is a vector space over $\mathbb{R}$.

**Example 1.4.** Let $U$ denote the set of all functions $f : [0,1] \longrightarrow \mathbb{R}$. Given $f, g \in U$ we may add the two function to get a function

$$(f+g) : [0,1] \longrightarrow \mathbb{R}$$

by setting

$$(f+g)(t) = f(t) + g(t).$$

Given a scalar $a \in \mathbb{R}$ and $f \in U$ we define a function $af \in U$ by setting

$$(af)(t) = a \cdot f(t).$$

This defines scalar multiplication. It is an exercise now to check that the above two operations makes $U$ into a vector space over $\mathbb{R}$.

**Example 1.5.** Let $U$ be as in the previous example and let $B \subseteq U$ denote the subset of $U$ consisting of those functions $f \in U$ that are bounded. In other words $f \in B$ if and only if there exists a constant $C$ such that

$$|f(t)| \leq C$$

for all $t \in [0,1]$. It is again an exercise to show that $B$ is a vector space over $\mathbb{R}$ with the same operations as in $U$.

**Example 1.6.** Let $U, B$ be as above. Let $V \subseteq U$ be the subset of those functions $f \in U$ that are continuous. Then as sum of of two continuous functions is continuous and scalar multiple of a continuous function is continuous we see that $V$ is also a vector space over $\mathbb{R}$. The following inclusions hold

$$V \subseteq B \subseteq U.$$

**Example 1.7.** Let $U$ be as above and let $W \subseteq U$ be the subset of $U$ consisting of those functions $f \in U$ for which

$$f(1/2) = 0.$$

That $W$ is a vector space over $\mathbb{R}$ with the same operations as in $U$ is left as an exercise.

The above examples lead us to the notion of a subspace. Here is the definition.

**Definition 1.8.** Let $V$ be a vector space over a field $F$. A subset $W \subseteq V$ is called a subspace of $V$ if $W$ is closed under addition and scalar multiplication.

In other words a subset $W$ of a vector space $V$ is a subspace if and only if it is also a vector space in its own right under the same operations as those in $V$.

We shall often use the shorthand $W \leq V$ to denote that $W$ is a subspace of $V$. In the above examples we have the following

$$V \leq B \leq U$$

and

$$W \leq U.$$

Here are some more examples.

**Example 1.9.** Let $V$ be a vector space over a field $F$. There are two canonical suabsaces of $V$. The subspaces $W = \{0\}$ consisting of just the zero vector in $V$ is clearly a subspace of $V$ and $W' = V$ is also a subspace. These are called the trivial subspaces of $V$. A subspace $W$ of $V$ is said to be proper if $W \neq \{0\}, V$.

**Example 1.10.** Let $V$ be a vector space over a field $F$ and $v \in V$ be a vector. Let

$$W = \{av \,:\, a \in F\}$$

We claim that $W$ is a subspace of $V$. To check this we need to check that $W$ is closed with respect to addition and scalar multiplication. So let $w_1, w_2 \in W$. Then $w_1 = a_1 v$ and $w_2 = a_2 v$ for some scalars $a_1, a_2 \in F$. As

$$w_1 + w_2 = a_1 v + a_2 v = (a_1 + a_2)v \in W$$

we conclude that $W$ is closed with respect to addition. If $a \in F$ is a scalar, then

$$a \cdot w_1 = a \cdot (a_1 v) = (aa_1)v \in W$$

shows that $W$ is closed with respect to scalar multiplication and hence is a subspace of $V$. The subsace $W$ is said to be spanned by the vector $v$.

**Definition 1.11.** Let $V$ be a vector space over a field $F$ and let $v_1 \dots v_k \in V$ be vectors. A vector $w$ which is a sum of the form

$$w = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = \sum_i a_i v_i \in V$$

where $a_1, \dots, a_k \in F$ are scalars is said to be a linear combination of the the vectors $v_1, \dots, v_k$.

**Remark 1.12.** We remark that the results we prove are valid for arbitrary fields unless otherwise stated. Most often our examples and exercises will use specific fields like $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$. We shall often just say $V$ is a vector space (instead of $V$ is a vector space over a field $F$) when the underlying field is not important or is understood from the context.

Here are some exercises.

**Exercise 1.13.** Check that the sets $U, B, V, W$ defined in Examples 1.4,1.5,1.6 and 1.7 are vector spaces over $\mathbb{R}$.

**Exercise 1.14.** Describe all proper subspaces of the vector space $\mathbb{R}^2$ of column vectors.

**Exercise 1.15.** Let $V$ be a vector space over a field $F$ and let $v_1, \dots, v_k \in V$ be vectors. Let $W$ be the subset of $V$ consisting of vectors that are linear combinations of the vectors $v_1, \dots, v_k$. Prove that $W$ is a subspace of $V$. $W$ is called the subspace spanned by the vectors $v_1, \dots, v_k$.

**Exercise 1.16.** Let $AX = 0$ denote a homogeneous system of equations with real coefficients. Prove that the set of solutions is a vector space over $\mathbb{R}$.