LINEAR ALGEBRA- LECTURE 11

1. Vector spaces

Recall that we had defined the notion of a vector space and looked at several examples. We look at two more before continuing with our discussion on vector spaces.

Example 1.1. Let F be a field and let F^n denote the set of all $n\times$ matrices

$$v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = (a_1, a_1, \dots, a_n)^t.$$

Then as we know, we may define addition and scalar multiplication in F^n so that F^n becomes a vector space over F.

Example 1.2. Let F be a field and let $M_{m\times n}(F)$ denote the set of $m \times n$ matrices $A = (a_{ij})$ with $a_{ij} \in F$. We now define addition of two such matrices by

$$(A_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \tag{1.2.1}$$

and scalar multiplcation by

$$r \cdot (a_{ij}) = (ra_{ij}). \tag{1.2.2}$$

One then easily checks that with these operations $M_{m\times n}(F)$ is a vector space over F.

Remark 1.3. It is important to note that, for example, in equation (1.2.1) the addition symbol $+$ appearing on either side of the equation are to be interpreted in a proper way. The addition symbol on the left hand side of equation (1.2.1) is the addition (that we are trying to define) in $M_{m\times n}(F)$ whereas the addition on the right side of the equation (1.2.1) is the one that exists in the field F. A similar consideration applies to the scalar product in equation (1.2.2).

Recall another definition that we introduced, namely, that of a subspace. Recall that a subset $W \subseteq V$ of a vector space V is called a subspace if W is closed with respect to addition and scalar multiplication of V. One way that subspaces arise is the following.

Example 1.4. Let V be a vector space of a field F and let $v \in V$ be a vector. Define

$$W = \{a, : a \in F\}.$$

Then it is easy to check that W is a subspace of V and we say that W is spanned by the vector v. That W is a subspace is left as an exercise.

Example 1.5. Let V be a vector space over F and let $v_1, \dots, v_k \in V$ be vectors. Recall that a vector $w \in V$ which is a sum of the form

$$v = a_1v_1 + \cdots + a_kv_k = \sum_i a_iv_i$$

is said to be a linear combination of the vectors $v_1, \ldots, v_k$. Let W be the subset of V consisting of linear combinations of the vectors $v_1, \ldots, v_k$, that is,

$$W = \left\{ v = \sum_i a_i v_i \ : \ a_i \in F \right\}.$$

It is an exercise to check that W is a subspace of W and we say that W is spanned by the vectors $v_1, \ldots, v_k$.

Here are examples of vector spaces that can be spanned by finitely many vectors.

Example 1.6. $\mathbb{R}$ is avector space over itself. It is spanned by any non-zzero vector. As a vector space over $\mathbb{R}$, $\mathbb{C}$ is spanned by the vectors $1, i$. If F is a field, then F^n is spanned by the column vectors $e_1, \ldots, e_n$.

Example 1.7. Let V denote the set of all polynomials with real coefficients in the variable x of degree less than n, $n \geq 1$. Then V is a vector space over $\mathbb{R}$ and is spanned by the vectors $1, x, \ldots, x^{n-1}$.

Example 1.8. Let $\mathbb{R}^\infty$ denote the set of all sequences $a = (a_1, a_2, \ldots)$ of real numbers such that there exists N (depending on a) with the property that $a_i = 0$ for all $i \geq N$. In other words a sequence $v = (a_1, a_2, \ldots)$ of real numbers belongs to $\mathbb{R}^{infty}$ if and only if the terms (or coordinates) of a are all zero after a certain stage. We may now give a vector space structure on $\mathbb{R}^\infty$ as follows. Addition is defined coordinatewise by

$$(a_1, a_2, \ldots) + (b_1, b_2, \ldots) = (a_1 + b_1, a_2 + b_2, \ldots)$$

and scalar multiplication by

$$r \cdot (a_1, a_2, \ldots) = (ra_1, ra_2, \ldots).$$

One then checks that these operations make $\mathbb{R}^\infty$ into a vector space over $\mathbb{R}$. Can we find finitely many vectors $v_1, v_2, \ldots v_k \in \mathbb{R}^\infty$ so that $\mathbb{R}^\infty$ is the span of the vectors $v_1, \ldots, v_k$?

We note the following easy observation.

Lemma 1.9. *Let V be a vector space over a field F and $W \subseteq V$ a subset. Then W is a subspace of V if and only if W is a vector space over F with the same operatiions as those of V.* □

Proof. Exercise. □

Thus a subspace of a vector space V is nothing but a subset that is also a vector space in its own right with the same operations as those in V. We shall often use the notation $W \leq V$ to mean that W is a subspace of the vector space V. Every vector space V has at least two subspaces, namely,

$$\{0\}, V \leq V.$$

These are called the trivial subspaces of V. A subspace $W \leq V$ is said to be proper if $W \neq \{0\}, V$.

We shall be dealing with an ordered an ordered set of vectors in a vector space. An ordered set of vectors will usually be denote by the round bracket as below. For example if $u, v \in V$ are vectors, then

$$S = (u, v)$$

with denote the ordered set of vectors where u appears first and v second. Thus

$$(u, v) \neq (v, u)$$

as ordered sets. Given an ordered set $S = (v_1, \dots, v_k)$ of vectors in a vector space V, $\text{span}(S)$ will denote the span of the ordered set S.

In what follows we shall always use the following convention. Given a vector space V over a field F and $v \in V$ and $a \in F$ we let

$$v \cdot = a \cdot v$$

denote the same vectors. At this point we note how the introduction of the notion of vector space connects with what we have studied so far, namely, the question of solutions to a system of equations. We first make a definition.

Definition 1.10. Let $A = (a_{ij})$ be a $m \times n$ matrix with entries in a field F. Let $A_1, A_2, \dots A_n$ denote the colums of the matrix A. Then each A_i is a column vector and

$$A_i \in F^m.$$

We then form the ordered set $S = (A_1, A_2, \dots, A_n)$. Then $\text{span}(S)$ is a subspace of F^m and is called the column space of the matrix A.

The following example shows how one can connect up the notion of vector spaces with system of linear equations.

Example 1.11. Let $AX = B$ denote a system of m linear equations in n variables where all the coefficients are in a field F. Then A is a $m \times n$ matrix, X is a $n \times 1$ column vetor of variables and $B \in F^m$ is a $m \times 1$ column vector. The product AX is a column vector which is actualy a sum of column vectors of the form

$$AX = A_1x_1 + A_2x_2 + \cdots + A_nx_n$$

where A_i is the i-th column vector of A. Thus $AX = B$ has a solution if and only if

$$B \in \text{span}(A_1, A_2, \dots A_n).$$

That is, $AX = B$ has a solution if and only if B belongs to the column space of the matrix A.

We shall now make a sequence of observations that will lead us to the notion of a basis and dimension of a vector space. We begin with the following observation.

Lemma 1.12. Let $S = (v_1, \dots, v_k)$ be an ordered set of vectors in a vector space V and $W \leq V$ a subspace. If $S \subseteq W$, then $\text{span}(S) \subseteq W$.

Proof. Let $v \in \text{span}(S)$, then $v = \sum a_i v_i$ for some $a_i \in F$. But as $v_i \in W$ and W is a subspace, we have

$$v = \sum_i v_i \in W.$$

□

Here are some problems.

Exercise 1.13. Check that in Examples 1.2, 1.4, 1.5, the sets in question are indeed vector spaces.

Exercise 1.14. Check that $\mathbb{R}^\infty$ cannot be spanned by finitely many vectors.

Exercise 1.15. Prove that the vector space V of all bounded functions $f : [0, 1] \longrightarrow \mathbb{R}$ cannot be spanned by finitely many functions.

Exercise 1.16. Assume the fact that every continuous function $f : [0, 1] \longrightarrow \mathbb{R}$ is bounded. Using the previous exercise show that the vector space V of all continuous functions $f : [0, 1] \longrightarrow \mathbb{R}$ cannot be spanned by finitely many functions.