LINEAR ALGEBRA- LECTURE 13

1. Vector spaces

Recall that last time we proved the following proposition.

Proposition 1.1. Let $S = (v_1, \ldots, v_m)$ be an ordered subset of a vector space V and let $w \in V$ be any vector. Let $S' = (L, w)$. Then

(1) $\text{span}(S) = \text{span}(S')$ if and only if $w \in \text{span}(S)$, and
(2) Let S be linearly independent. Then S' is linearly independent if and only if $w \notin \text{span}(S)$.

Suppose that $B = (v_1, \ldots v_m)$ is a basis of a vector space V over a field F. Then given a vector $v \in V$ we may write

$$v = \sum_i a_i v_i$$

where $a_i \in F$. If v can also be written as

$$v = \sum_i b_i v_i$$

then the relation

$$\sum_i (a_i - b_i) v_i = 0$$

forces $a_i = b_i$. Thus, if $B = (v_i, \ldots, v_n)$ is a basis of V, then each vector $v \in V$ is a linear combination of v_i in a unique way.

Recall that a vector space V over a field F is finite dimensional if there exists a finite set $S = (v_1, \ldots, v_n)$ such that

$$\text{span}(S) = V.$$

In other a words a vector space is finite dimensional if it can be spanned by finitely many vectors.

Proposition 1.2. Let V be a finite dimensional vector space over a field F.

(1) Let $S = (v_1, \ldots, v_m)$ span V and let L be an independent subset f V. Then one can get a basis by adding (suitable) elements of S to L.
(2) Let $S = (v_1, \ldots, v_m)$ span V. Then one can get a basis of V by deleting (suitable) elements of S.

Proof. We first prove (1). Let i_1 be the smallest integer, $1 \le i_1 \le m$ such that $v_{i_1} \notin \text{span}(L)$. If no such index exists then L spans V and hence is a basis. Then by Proposition 1.1,

$$L_1 = (L, v_{i_1})$$

is linearly independent and

$$v_i \in \text{span}(L_1)$$

for all i, $1 \le i \le i_i$. Next let i_2, $i_1 < i_2 \le m$ be the smallest integer such that $v_{i_2} \notin \text{span}(L_1)$ and set

$$L_2 = (L_1, v_{i_2}).$$

Then by Proposition 1.1, L_2 is linearly independent and $v_i \in \mathrm{span}(L_2)$, $1 \le i \le i_2$. In finitely many steps we get a linearly independent set L' such that

$$S \subseteq \mathrm{span}(L').$$

This means

$$V = \mathrm{span}(S) \subseteq \mathrm{span}(L')$$

and therefore $V = \mathrm{span}(L')$ and so L' is a basis. This completes the proof of (1).

We next prove (2). If S is linearly independent we are done. If not we can find $a_1, a_2, \ldots, a_m \in F$ not all zero such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0.$$

Let i_i be the smallest integer $1 \le i_1 \le m$ such that $a_{i_1} \neq 0$. Consider the set S_1 obtained from S by deleting v_{i_1}, that is

$$S_1 = (v_1, \ldots, \widehat{v_{i_1}}, \ldots, v_m).$$

Now as

$$v_{i_1} = -(1/a_{i_1}) \sum_{i \neq i_1} a_i v_i$$

we see that $v_{i_1} \in \mathrm{span}(S_1)$ and hence by Proposition 1.1 we have

$$\mathrm{span}(S_1) = \mathrm{span}(S) = V.$$

If S_1 is linearly independent we are done. Otherwise, as before, there is a relation

$$\sum_{i \neq i_i} a_i v_i = 0$$

and we let i_2 be the smallest integer, $i_1 < i_2 \le m$ such that $a_{i_2} \neq 0$. Proceeding as before we let S_2 to be the set obtained from S_1 by deleting v_{i_2} and observe that

$$\mathrm{span}(S_2) = \mathrm{span}(S_1) = \mathrm{span}(S) = V.$$

Proceeding this way we obtain a set S' that is now linearly independent and such that $\mathrm{span}(S') = V$. This completes the proof of (2). □

We need to be careful in the proof of case (2) in the above proposition. For example what can happen is that all the vectors v_i could be the xero vectors and we will then be forced to throw out every vector and then $S' = \emptyset$. To resolve this we agree that the empty set is linearly independent and that the span of the empty set of vectors is the vector space $\{0\}$.

Remark 1.3. It is important to note the following interpretation of the two cases of the above proposition. The case (1) says that any linearly independent set L in a finite dimensional vector space V may be enlarged to a basis by adjoining suitable vectors. All we have to do is fix a spanning set (which exists since V is finite dimensional) and pick suitable vectors from the spanning set and adjoin these to L. The second equally important point made by case (2) is that every spanning set S contains a basis.

Theorem 1.4. Let $S = (v_1, \ldots, v_m)$ span the vector space V and let $L = (w_1, \ldots, w_n)$ be an linearly independent set. Then $m \ge n$

Proof. Notice that V is finite dimensional. The theorem says that (in a finite dimensional vector space) the number of elements in any spanning set is greater than or equal to the number of elements in any linearly independent set. We shall think of S and L as a row matrix (of vectors) and although multiplication of matrices of vectors does not make sense we may however right multiply by column matrices

$$LX = (w_1, \ldots, w_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = v_1x_1 + \cdots + v_nx_n.$$

Here $x_i \in F$. Thus as L is a linearly independent set and x_i are variables, then the equation

$$LX = 0 \tag{1.4.1}$$

has only yhe trivial solution.

We now prove the theorem by contradiction. Assume, if possible, that $m < n$. As S spans V, we may write each w_j as

$$w_j = v_1a_{1j} + v_2a_{2j} + \cdots + v_ma_{mj}.$$

This can be expressed as the product

$$w_j = (v_1, \ldots, v_m) \begin{pmatrix} a_{1j} \\ \vdots \\ v_{mj} \end{pmatrix} = SA_j.$$

If A denote the matrix whose j-th column is A_j we then have

$$SA = (SA_1, \ldots, SA_n) = (w_1, \ldots, w_n) = L.$$

Thus

$$(SA)X = S(AX) = LX.$$

The associativity in the above equation can be readily checked. Now if $AX = 0$, then $S(AX) = 0$ and therefore $LX = 0$. But as $m < n$, the equation $AX = 0$ has a non zero solution which means that $LX = 0$ has a non zero solution which is a contradiction. Thus $m \geq n$. □

It is important to understand what the above theorem says. The above theorem has important consequences. We now study some of the consequences.

Corollary 1.5. Let V be a finite dimensional vector space. Then any two bases have the same number of elements.

Proof. Let $B_1 = (v_1, \ldots, v_m)$ and $B_2 = (w_1, \ldots, w_n)$ be two bases of V. Then as B_1 spans V we have $m \geq n$. Similarly $n \geq m$ and equality follows. □

Corollary 1.6. Let B be a basis of the finite dimensional vector space V and let S span V. Then $|S| \geq |B|$. Further $|S| = |B|$ if and only if S is a basis.

Proof. That $|S| \geq |B|$ is a consequence of Theorem 1.4. If S is a basis, then $|S| = |B|$ follows from Corollary 1.5. Conversely assume that $|S| = |B|$. Then by Proposition 1.2 we may delete elements of S to obtain a basis of V. But then we will get a set with fewer elements than $|B|$ which is a basis. This contradicts Corollary 1.5. Thus S must itself be a basis. □

Corollary 1.7. Let B be a basis of the finite dimensional vector space V and let L be an independent subset of V. Then $|L| \leq |B|$ and equality holds if and only L is a basis.

Proof. Since any independent set can be enlarged to a basis and any two bases have the same number of elements we have that $|L| \leq |B|$. Suppose that $|L| = |B|$ holds. If L is not a basis, then again as it can be enlarged to a basis we would get a basis of larger size. Hence L is a basis. □

Observe that any finite dimensional vector space has a basis. This is a consequence of Proposition 1.2. One now makes the following definition.

Definition 1.8. The dimension of a finite dimensional vector space V is the number of element in any basis of V. This is denote by $\dim(V)$.

Here are some examples.

Example 1.9. $\mathbb{R}$ is one-dimensional as a vector space over itself. $\mathbb{C}$ is two-dimensional as a vector space over $\mathbb{R}$.

Example 1.10. If F is a field, then F^n is n-dimensional over F.

Example 1.11. Let V denote the vector space of all polynomials of degree at most n with real coefficients. Then V is a vector space over $\mathbb{R}$ of fimension $n+1$.

Proposition 1.12. Let W be a subspace of a finite dimensional vector space V. Then W is finite dimensional and $\dim(W) \leq \dim(V)$ and equality holds if and only if $W = V$.

Proof. Suppose L is an independent (and therefore a finite) subset of W. If L spans W, then L is a basis of W and hence W is finite dimensional. Since L is linearly independent, it stays an independent set when viewed as a subset of V. Now L can be enlarged to a basis of V and hence $\dim(W) \leq \dim(V)$ also holds.

If L does not span W, we find $w \in W$ such that $w \notin \operatorname{span} L$ and let $L' = (L, w)$. Then L' is independent as a subset of W. Since it is also independent as a subset of V we have, by Theorem 1.5, that $|L| \leq \dim(V)$. Now if L' spans W, we are done otherwise we construct a larger set $L'' = (L', w')$ that is linearly independent and $|L'| \leq \dim(V)$. After finitely many steps we get an independent subset L_1 of W that spans W and $|L_1| \leq \dim(V)$. This shows that W is finite dimensional and that

$$\dim(W) \leq \dim(V).$$

If $|L_1| = \dim(V)$, then by Corollary 1.7, L is a basis of V and hence $W = V$. □

Here are two examples.

Example 1.13. Let V be the vector space over $\mathbb{R}$ of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$. The addition and scalar multiplication is defined pointwise. Consider the vectors $u = x^2, v = \cos x, w = e^x$. We claim that these three functions are linearly independent. Consider the relation

$$ax^2 + b\cos x + ce^x = 0.$$

Differentiating thrice we obtain

$$b \sin x = -ce^x. \tag{1.13.1}$$

Since e^x is never zero and (1.13.1) holds for all x we conclude that $b = c = 0$. Hence $a = 0$.

A somewhat unrelated example.

Example 1.14. Recall that F_2 is the field with two elements $F_2 = \{0, 1\}$ where addition and multiplication is defined by

$$0 + 1 = 0 = 1 + 0, \ \ 1 + 1 = 0$$
$$1 \cdot 0 = 0 \cdot 1 = 0, \ \ 1 \cdot 1 = 1.$$

Thus 0 is the additive identity and the additive inverse of 1 is itself, that is, $-1 = 1$. Also note that for any integer n, $n \cdot 1 = 0$ if n is even and $n \cdot 1 = 1$ if n is odd. We now wish to solve the system of equations

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{1.14.1}$$

This is an equation with coefficients in F_2 since we will think of -1 as 1. To solve this system we form the augmented matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

and row reduce it to the row echelon form as below.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \xrightarrow{X_1:X_2} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \xrightarrow{X_2:X_2+X_1} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{X_3:X_3+X_1} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \xrightarrow{X_3:X_3+X_2} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{X_2:X_2+X_3} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which is now in the row echelon form. Thus the original system of equations has the same solutions as the system

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \tag{1.14.2}$$

The solutions can now readily be read off as $x_1 = -1 = 1, x_2 = 0, x_3 = 0$.

Here are some problems.

Exercise 1.15. Decide whether the system (1.14.1) has solution over the rational numbers.

Exercise 1.16. Solve the system of equations

$$\begin{pmatrix} 6 & -3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

over F_2.

Exercise 1.17. Let 0 be a scalar and $v \in V$ a vector. Show that the scalar product $0 \cdot v$ equals the zero vector in V.

Exercise 1.18. Decide which of the following subsets of $M_n(\mathbb{R})$ are subspaces.

$$W = \{A \in M_n(\mathbb{R}) \,:\, A = A^t\}$$
$$V = \{A \in M_n(\mathbb{R}) \,:\, A \text{ is invertible}\}.$$
$$U = \{A \in M_n(\mathbb{R}) \,:\, A \text{ is upper triangular}\}.$$

Exercise 1.19. Find a basis for the set of $n \times n$ matrices A with $A = A^t$. Such matrices are called symmetric matrices.

Exercise 1.20. Let A be a $m \times n$ matrix and let A' be obtained from A by a sequence of row operations. Show that the rows of A and the rows of A' span the same space.

Exercise 1.21. Let $V = F^n$ be the space of column vectors. Prove that every subspace W of V is the space of solutions of some system of homogeneous linear equations $AX = 0$.