LINEAR ALGEBRA- LECTURE 14

1. Linear transformations

A linear transformation is a special type of function between vector spaces, one that respects the vector space structure. More precisely we have the following definition.

Definition 1.1. Let V, W be two vector spaces over the (same) field F. A function $f : V \longrightarrow W$ is said to be a linear transformation (or a linear map or simply linear) if

$$f(u+v) = f(u) + f(v), \quad f(au) = af(u)$$

for all $u, v \in V$ and $a \in F$.

Note that for a function between vector spaces to be a linear transformation both the vector spaces have to be vector spaces over the same field. Thus in the statement : Let $f : V \longrightarrow W$ be alinear map....it is implicit that f is a linear transformation where V, W are vector spaces over the same field. Here are some examples.

Example 1.2. Let V, W be vector spaces over F. Then the map $f : V \longrightarrow W$ defined by $f(v) = 0$ for all $v \in V$ is linear. f is called the zero linear transformation.

Example 1.3. Fix $a \in \mathbb{R}$. Thinking of $\mathbb{R}$ as a vector space over itlself, the map $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = ax$ for all $x \in \mathbb{R}$ is a linear map.

Example 1.4. Let $1 \le k \le n$. Consider the map

$$f : M_{m\times n}(\mathbb{R}) \longrightarrow \mathbb{R}^m$$

defined by

$$f\left((a_{ij})\right) = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{pmatrix}$$

is a linear map for each k. The map g defined by

$$g\left((a_{ij})\right) = a_{11}$$

gives a linear map $g : M_{m\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$. Here $M_{m\times n}(\mathbb{R})$ and $\mathbb{R}$ are thought of as vector spaces over $\mathbb{R}$.

Example 1.5. Let V denote the set of all continuous functions $f : [0,1] \longrightarrow \mathbb{R}$. Then we know V is a vector space over $\mathbb{R}$ with respect to pointwise addition and scalar multiplication. The map $T : V \longrightarrow \mathbb{R}$ defined by

$$T(f) = \int_0^1 f(x)\, dx$$

is a linear map.

Example 1.6. For $n \geq 0$ let $P_n(x)$ denote the set of all polynomials $p(x)$ with real coefficients of degree at most n. The map $f : P_{n+1}(x) \longrightarrow P_n(x)$ defined by

$$f(p(x)) = \frac{d}{dx}p(x) = p'(x)$$

is a linear map.

Example 1.7. Let V be a finite dimensional vector space over a field F. Fix a basis $B = (v_1, v_2, \dots, v_n)$ of V. Then each $v \in V$ can be uniquely expressed as

$$v = a_1v_1 + \cdots + a_nv_n \tag{1.7.1}$$

for some $a_i \in F$. Consider the map

$$f : V \longrightarrow F^n$$

defined by

$$f(v) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Then f is a linear map. It is clear that the function f is onto. The uniqueness of the expression in (1.7.1) implies that f, in addition, is also $1-1$

The above example leads us to the following definition.

Definition 1.8. Let V, W be two vector spaces over the field F. A linear map $f : V \longrightarrow W$ is said to be an isomorphism if, in addition, f is both $1-1$ and onto. If such a map exists we say that V and W are isomorphic and we write $V \cong W$.

For example, the map f in Example 1.7 is an isomorphism and thus if V is a n-dimensional vector space ver F, then

$$V \cong F^n.$$

When two vector spaces are isomorphic we then think of the two vector spaces to be the same in all respects.

Remark 1.9. Often in mathematics we encounter sets with similar structure (for example two vector spaces) and wish to compare them. One way is to use functions between the two sets. However to keep track of the additional structure present, one is interested in those functions that respect the structure. That lead us to the definition of a linear transformations between vector spaces. While studying sets with additional structure one is also interested in trying to understsnd when two such structure are the "same". That is the notion of isomorphism of vector spaces.

Remark 1.10. A word about some alternative terminology. A linear transformation $f : V \longrightarrow W$ which is also $1-1$ is also said to be a monomorphism. Thus

$$\text{monomorphism} = \text{linear} + (1-1).$$

A linear map $f : V \longrightarrow W$ that is also onto is said to be an epimorphism. Thus

$$\text{epimorphism} = \text{linear} + \text{onto}.$$

Here are some problems.

Exercise 1.11. Check that the maps defiend in Examples 1.1-1.7 are all linear.

Exercise 1.12. Construct an isomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ of real vector spaces.

Exercise 1.13. Show that $P_n(x) \cong \mathbb{R}^{n+1}$.

Exercise 1.14. Show that every vector space V over $\mathbb{C}$ is also a vector space over $\mathbb{R}$.

Exercise 1.15. Let V be a 1-dimensional vector space over $\mathbb{C}$. Find the dimension of V over $\mathbb{R}$. Generalize.

Exercise 1.16. Show that the composition of two linear maps is again linear. Furher show that the composition of two isomorphisms is an isomorphism. Conclude that any two vector spaces (over the same field) of the the same dimension are isomorphic.

Exercise 1.17. Show that the inverse of a vector space isomorphism is an isomorphism.

Exercise 1.18. Consider the map

$$f : P_1(x) \longrightarrow \mathbb{R}^2$$

defined by

$$f(a + bx) = \begin{pmatrix} a - b \\ b \end{pmatrix}.$$

Is f an isomorphism?

Exercise 1.19. Let V denote the vector space over $\mathbb{R}$ of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$. Define an ordered set S of vectors in V by setting

$$S = (\sin\theta, \cos\theta).$$

Let $W = \operatorname{span}(S)$ be the span of S. Then W is a subspace of V. W is isomorphic to which familiar vector space?

Exercise 1.20. Fix $a \in \mathbb{R}$. Consider the map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = a \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ ay \end{pmatrix}.$$

Show that f is an isomorphism if and only if $a \neq 0$. Generalize. This isomorphism of $\mathbb{R}^2$ with itself tells us more than the obvious fact that a vector space is isomorphic to itself. It tells us more about the nature of the isomorphisms possible. In this example dilation. Another example is given in the next exercise.

Exercise 1.21. Consider the map $f : P_5(x) \longrightarrow P_5(x)$ defined by

$$f(p(x)) = p(x - 1).$$

Show that f is an isomorphism.

Exercise 1.22. Exhibit subspaces W_1, W_2 of $\mathbb{R}^3$ such that $W_i \cong \mathbb{R}^2$ for $i = 1, 2$. In your example, compute $W_1 \cap W_2$.

Exercise 1.23. Describe all isomorphisms $f : \mathbb{R} \longrightarrow \mathbb{R}$.