LINEAR ALGEBRA- LECTURE 15

1. Change of basis

Throughout this section we fix a field F and all vector spaces that we consider are vector spaces over F. Let $S = (v_1, v_2, \ldots, v_n)$ be an ordered subset of a vector space V. We often think of

$$S = (v_1, \ldots v_n)$$

as a $1 \times n$ row vector (also called a hypervector). Given a $n \times 1$ column vector

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in F^n$$

the multiplication SX is defined to be

$$SX = (v_1, \ldots, v_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = v_1x_1 + v_2x_2 + \cdots + v_nx_n.$$

and is thus a linear combination of $v_1, \ldots, v_n$. More generally the product SA is defined where A is a $n \times n$ matrix and the product is then a $1 \times n$ hypervector. This product is defined as follows

$$SA = (v_1 \ldots, v_n) \begin{pmatrix} | & \ldots & | \\ A_1 & \ldots & A_n \\ | & \ldots & | \end{pmatrix} = (SA_1, \ldots, SA_n).$$

Here, as usual, A_i denotes the i-th column of A.

Next we recall the following. Suppose $B = (v_1, \ldots, v_n)$ is a basis of the vector space V. Given a vector $v \in V$, we may write v uniquely as

$$v = a_1v_1 + x_2v_2 + \cdots + x_nv_n$$

where $a_i \in F$. The scalars a_i are called the coordinates of v and the column vector

$$X = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

is called the coordinate vector of v. Note that v may also be written as the matrix product

$$v = BX. \tag{1.0.1}$$

We now consider the situation where we are given two bases

$$B = (v_1, \ldots, v_n)$$
$$B' = (v'_1, \ldots, v'_n)$$

of the vector space V. We may write each v'_j uniquely as

$$v'_j = v_1 p_{1j} + v_2 p_{2j} + \cdots + v_n p_{nj}$$

where $p_{ij} \in F$. Keeping (1.0.1) in mind we have that

$$v'_j = (v_1, \ldots, v_n) \begin{pmatrix} p_{1j} \\ \vdots \\ p_{nj} \end{pmatrix} = BP_j.$$

where P_j is the obvious column vector. Thus if P is the $n \times n$ matrix whose j-th row is P_j we have

$$(v'_1, \ldots, v'_n) = (v_1, \ldots, v_n) \begin{pmatrix} | & \ldots & | \\ P_1 & \ldots & P_n \\ | & \ldots & | \end{pmatrix} = (BP_1, \ldots, BP_n).$$

and hence

$$B' = BP.$$

In other words if we are given bases B, B' of a vector space V, then there exists a $n \times n$ matrix P such that

$$B' = BP.$$

Definition 1.1. The matrix P constructed above is called the basechange matrix from B to B'.

Before looking at examples we prove some basic properties of the basechange matrix.

Proposition 1.2. Let B, B' be two bases of the finite dimensional vector space V. Let P be the basechange matrix from B to B'. Then P is invertible.

Proof. Since P is the basechange matrix from B to B' we have the basic equality

$$B' = BP.$$

Let Q denote the basechange matrix from B' to B. Then

$$B = B'Q.$$

Thus

$$B = B'Q = B(PQ).$$

This may be rewritten in the form

$$(v_1, v_2, \ldots, v_n) = (v_1, v_2, \ldots, v_n)(PQ).$$

This clearly forces $PQ = I$ and hence P is invertible. □

The above proposition says that if B, B' are two bases of the finite dimensional vector space V, then we can find an invertible matrix P such that the basis B' is got from the matrix multiplication BP. This observation infact allows us to say something more.

Proposition 1.3. Suppose B is a basis of the finite dimensional vector space V. The every basis B' of V if of the form

$$B' = BP$$

for some invertible matrix P.

Proof. If B' is any other basis and P is the basechange matrix from B to B', then by the previous proposition we have

$$B' = BP$$

where P is invertible. Conversely suppose that P is an invertible matrix and B' is the ordered set of vectors given by

$$B' = BP.$$

We claim that B' is a basis of V. Let $B = (v_1, \ldots, v_n)$ and let $B' = (v'_1, \ldots, v'_n)$. Now B spans V. To show that B' spans V it is enough to check that each v_i is a linear combination of $v'_1, \ldots, v'_n$. But this follows from the equality

$$(v_1, \ldots, v_n) = (v'_1, \ldots, v'_n)P^{-1}.$$

Now as B' spans V and has the same number of elements as B it follows that B' is a basis of V. □

The above proposition in effect says that if we know one basis B of the finite dimensional vector space V, then we know all the bases. How? We look at all the ordered sets B' that we get as the product

$$B' = BP$$

by letting P vary over all possible invertible matrices. Then each such B' is a basis and all bases are of this form.

Next we consider the following situation. Suppose B, B' are two bases of the finite dimensional vector space V. Let $v \in V$ and let $X, X' \in F^n$ be the coordinate vectors of v with respect o the bases B, B' respectively. We wish to understand how X and X' are related. First, let P be the base change matrix from B to B' so that

$$B' = BP. \tag{1.3.1}$$

Now as X, X' are the coordinate vectors of v relative to the bases B, B' we have that

$$v = BX, \quad v = B'X'.$$

Thus

$$v = BX = B'P^{-1}X = B'(P^{-1}X). \tag{1.3.2}$$

Thus we have two ways of writing v as a linear combination of the basis B' :

$$v = B'X', \quad v = B'(P^{-1}X).$$

But since any such expression of v as a linear combination of basis elements is unique we must have

$$X' = P^{-1}X$$

and therefore

$$PX' = X. \tag{1.3.3}$$

The relations (1.3.1) and (1.3.3) are fundamental. Here is an example.

Example 1.4. Let W be the set of solutions of the equation

$$2x_1 - x_2 - 2x_3 = 0.$$

with real coefficients. We then know that W is a subspace of $\mathbb{R}^3$. It is easy to argue that

$$\dim(W) \leq 2.$$

That $\dim(W) = 2$ follows from the observation that

$$B = \big(v_1 = (1,2,0)^t, v_2 = (0,-2,1)^t\big)$$
$$B' = \big(v'_1 = (2,2,1)^t, v'_2 = (1,0,1)^t\big)$$

are linearly independent subsets of W and hence must be bases of W. We wish to compute the base change matrix from B to B'. To compute the basechange matrix we observe that

$$v_1' = 2v_1 + v_2$$

and

$$v_2' = v_1 + v_2.$$

By definition, the 2×2 matrix

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

is the basechange matrix from B to B'. We may now apply the equation (1.3.3) to compute the coordinate vector of any $w \in W$ relative to B if we know its coordinate vector relative to B' and vice versa.

Exercise 1.5. Let $B = (e_1, e_2)$ denote the standard basis of $\mathbb{R}^2$. Let $B' = (u, v)$ be any basis of $\mathbb{R}^2$. Compute the base change matrix from B to B'. Compute the base change matrix from B' to B.

Exercise 1.6. Find a basis of the space of solutions of the equation

$$x_1 + 2x_2 + \cdots + nx_n = 0$$

in $\mathbb{R}^n$.

Exercise 1.7. Let $B = (v_1, \ldots, v_n)$ be an ordered set of vectors in F^n. Show that B is a basis of F^n if and only if the $n \times n$ matrix

$$\begin{pmatrix} | & \cdots & | \\ & & \\ v_1 & \cdots & v_n \\ & & \\ | & \cdots & | \end{pmatrix}$$

is invertible. Note that the i-th column of the above matrix if the column vector v_i.

Exercise 1.8. Prove that the set $B = ((1,2,0)^t, (2,1,2)^t, (3,1,1)^t)$ is a basis of $\mathbb{R}^3$. Find the coordinate vector of the vector $(1,2,3)$ with repect to this basis. Let B' denote the standard basis of $\mathbb{R}^3$. Find the base change matrix from B to B'.

Exercise 1.9. Let $B = (e_1, e_2)$ denote the standard basis of $\mathbb{R}^2$. Show that $B' = (e_1 + e_2, e_1 - e_2$ is a basis of $\mathbb{R}^2$. Find the base change matrix from B to B'

Exercise 1.10. Let $B = (e_1, e_2, \ldots, e_n)$ denote the standard basis of $\mathbb{R}^n$. Find the basechange matrix from B to B' where $B' = (e_n, e_{n-1}, \ldots, e_1)$.