LINEAR ALGEBRA- LECTURE 16

1. Direct sums

In this section we study a new way of constructing new vector spaces, namely, the notion of the direct sum of vector spaces. We begin with the definition of the sum of vector spaces.

Definition 1.1. Let $W_1, \dots, W_k$ be subspaces of the vector space V. The set

$$W_1 + \cdots + W_k = \{v \in V \,:\, v = w_1 + \cdots + w_k, w_i \in W_i\}$$

is clearly a subspace of V and is called the sum of the vector spaces $W_1, \dots, W_k$. This will sometimes be denoted by $\Sigma_i W_i$.

Thus $W_1 + \cdots + W_k$ consists precisely of those vector $v \in V$ that can be represented as

$$v = w_1 + \cdots + w_k$$

a sum of k many vectors $w_1, \dots, w_k$ with $w_i \in W_i$. The proof that $\Sigma_i W_i$ is a subspace of V is left as an exercise. Here are two examples.

Example 1.2. As usual let $e_1 = (1, 0)^t \in \mathbb{R}^2$. Let $W_1 = \mathrm{span}(e_1)$ and $W_2 = \mathrm{span}(-e_1)$ be subspaces of $\mathbb{R}^2$. It can now be checked that

$$W_1 + W_2 = W_1 = W_2.$$

Example 1.3. Let $e_1 = (1, 0)^t, v = (1, 1)^t \in \mathbb{R}^2$. Let $W_1 = \mathrm{span}(e_1)$ and $W_2 = \mathrm{span}(v)$ be subspaces of $\mathbb{R}^2$. It can now be checked that

$$W_1 + W_2 = \mathbb{R}^2.$$

Let us try to understand some properties of the sum of vector spaces. We first make a definition.

Definition 1.4. The subspaces $W_1, \dots, W_k$ of a finite dimensional vector space V are said to be independent if whenever we have

$$w_1 + w_2 + \cdots + w_k = 0$$

with $w_i \in W_i$, then $w1 = w_2 = \cdots = w_k = 0$.

For example the subspaces W_1, W_2 in Example 1.2 are not independent whereas the ones in Example 1.3 are independent. This can be readily verified. Here is another example.

Example 1.5. Let W_1, W_2 be subspaces of a finite dimensional vector space V. Assume that W_1, W_2 are independent. Let $0 \neq v \in W_1 \cap W_2$. Since $v \in W_2$ we must have $-v \in W_2$. Now if $w_1 = v$ and $w_2 = -v$, then

$$w_1 + w_2 = 0$$

but neither of w_1, w_2 is zero contradicting the fact that W_1, W_2 are independent. Thus $v = 0$. Consequently, if W_1, W_2 are independent, then

$$W_1 \cap W_2 = \{0\}.$$

Conversely suppose that $W-1 \cap W_2 = \{0\}$. We shall show that W_1, W_2 are independent. So let there be a relation

$$w_1 + w_2 = 0$$

with $w_i \in W_i$, $i = 1, 2$. Thus $w_1 = -w_2$ and hence $w_2 \in W_1$. Thus $w_2 \in W_1 \cap W_2$ we must have $w_2 = 0$ and hence $w_1 = 0$ which shows that W_1, W_2 are independent. Thus two subspaces W_1, W_2 of V are independent if and only if $W_1 \cap W_2 = \{0\}$.

Proposition 1.6. Let $W_1, \ldots, W_k$ be subspaces of a finite dimensional vector space V and let B_i be a basis of W_i for $i = 1, 2, \ldots, k$. Then the following statements are equivalent.

(1) $W_1, \ldots, W_k$ are independent and $W_1 + \cdots + W_k = V$.
(2) The set $B = (B_1, \ldots, B_k)$ is a basis of V.

Proof. Assuming that (1) holds we check that B is a basis of V. Let $v \in V$. Then we may write

$$v = w_1 + \cdots + w_k$$

with $w_i \in W_i$. Since each w_i can be written as a linear combination of the vectors in B_i $(1 \leq i \leq k)$ it follows that v can be written as a linear combination of the vectors in B. This shows that B spans V. Next assume that $w = 0$ where w is a linear combination of vectors in B. We may then write

$$w = \sum_i w_i = 0$$

where each

$$w_i = \sum_j a_{ij} v_{ij}$$

is a linear combination of vectors in B_i and where for each i, the vectors v_{ij} are the basis vectors in B_i. But as $W_1, \ldots, W_k$ are independent we must have each $w_i = 0$. This forces $a_{ij} = 0$ for all i, j. This shows B is linearly independent and hence a basis of V.

Conversely assume that (2) holds. Then as B spans V we clearly have that

$$W_1 + \cdots + W_k = V.$$

So all that remains to be checked is that $W_1, \ldots, W_k$ are independent. So assume that there is a relation

$$w_1 + \cdots + w_k = 0$$

where $w_i \in W_i$. Then we may write

$$w_i = \sum_j a_{ij} v_{ij}.$$

It now follows, since B is independent, that $a_{ij} = 0$ for all i, j. Thus $w_i = 0$ for all i. This completes the proof. $\square$

Proposition 1.7. Suppose $W_1, \ldots, W_k$ are subspaces of the finite dimensional vector space V. Then

$$\dim(W_1 + \cdots + W_k) \leq \dim(W_1) + \cdots + \dim(W_k). \tag{1.7.1}$$

Equality holds if and only if $W_1, \ldots, W_k$ are independent.

Proof. Let B_i be a basis of W_i, $i = 1, \ldots, k$ and assume that $|B_i| = n_i$, $i = 1, \ldots, k$. Then $B = (B_1, B_2, \ldots, B_k)$ spans $W_1 + \cdots + W_k$ and hence

$$\dim(W_1 + \cdots + W_k) \leq |B| = n_1 + \cdots + n_k = \dim(W_1) + \cdots + \dim(W_k).$$

This proves first part of the proposition. Suppose equality holds in (1.7.1), then $B = (B_1, \ldots, B_k)$ is a basis of $\sum_i W_i$. By the previous proposition we have that $W_1, \ldots, W_k$ are independent. On the other hand if $W_1, \ldots, W_k$ are independent then, again by the previous proposition, $B = (B_1, \ldots, B_k)$ is a basis of $\sum_i W_i$ and hence the equality in (1.7.1) must hold. This completes the proof. $\square$

We end this section with a definition.

Definition 1.8. Let $W_1, \ldots, W_k$ be subspaces of V. We say that V is the direct sum of W_1, W_2 if $V = W_1 + \cdots + W_k$ and $W_1, \ldots, W_k$ are independent. If this happens we write

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

2. Infinite dimensional spaces

Recall that a vector space V is finite dimensional if there exists a finite subset S of V such that

$$\operatorname{span}(S) = V.$$

If no such S exists we say that V is infinite dimensional. Here are some examples.

Example 2.1. Let $\mathbb{R}^\infty$ denote the set of all sequences $(a_1, a_2, \ldots)$ of real numbers that are eventually zero. In other words a sequence $(a_1, a_2, \ldots) \in \mathbb{R}^\infty$ if and only if there exists $N > 0$ such that $a_i = 0$ for all $i \geq N$. Of course this N will be different for different elements of $\mathbb{R}^\infty$. We know that $\mathbb{R}^\infty$ is a vector space under coordinate wise addition and scalar multiplication. We have also seen that $\mathbb{R}^\infty$ is not finite dimensional.

Example 2.2. Let $\mathbb{R}^\omega$ denote the set of all sequences $(a_1, a_2, \ldots)$ of real numbers. Then $\mathbb{R}^\omega$ is a vector spaces under coordinate wise addition and scalar multiplication. In fact,

$$\mathbb{R}^\infty \leq \mathbb{R}^\omega$$

and hence $\mathbb{R}^\omega$ is infinite dimensional.

Example 2.3. Let V denote the set of all polynomials in the variable x with real coefficients. Then with the usual definition of addition and scalar multiplication of polynomials we have that V is a vector space over $\mathbb{R}$. Then it is an exercise to check that V is infinite dimensional over $\mathbb{R}$.

Example 2.4. Let V denote the set of all continuous functions $f : [0, 1] \longrightarrow \mathbb{R}$. Then V is a vector space over $\mathbb{R}$ and is infinite dimensional.

We can make sense of the span of an infinite subset of a vector space. Here is the definition.

Definition 2.5. Let S be a subset of a vector space V. Then the span of the set S, denoted by $\operatorname{span}(S)$ is by definition the set of all linear combinations of finitely many elements of S.

For example let e_i denote the sequence

$$e_i = (0, \ldots, 0, 1, 0, \ldots) \in \mathbb{R}^\infty$$

where 1 is in the i-th position. Then if $S = (e_1, e_2, \ldots)$ it is clear that

$$\operatorname{span}(S) = \mathbb{R}^\infty.$$

One may now define linear independence of a (possibly infinite) subset of a vector space.

Definition 2.6. Let S be a subset of a vector space V. We say that S is linearly independent if every finite subset is linearly independent.

Finally a basis is defined as follows.

Definition 2.7. A subset B of a vector space V is said to be a basis if B spans V and is linearly independent.

For example the set $S = (e_1, e_2, \ldots)$ discussed above is a basis of $\mathbb{R}^\infty$. Finally we note an important fact without proof.

Theorem 2.8. Every vector space has a basis. □

Here are some problems.

Exercise 2.9. Complete the proof of the claims made in Examples 1.2-1.3.

Exercise 2.10. Show that the vector space V in Examples 2.3-2.4 is infinite dimensional.

Exercise 2.11. Let W_1, W_2 be subspaces of a finite dimensional vector space V. Show that

$$\dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2).$$

Exercise 2.12. Let $W_1 + W_2 = V$. Show that there is a subspace W_2' of W_2 such that

$$V = W_1 \oplus W_2'.$$

Exercise 2.13. Let V be a finite dimensional vector space and W a subspace. Does there always exist a subspace W' of V such that $V = W \oplus W'$?

Exercise 2.14. A $n \times n$ matrix A is said to be symmetric if $A^t = A$ and skew-symmetric if $A^t = -A$. Let $\mathrm{Sym}_n(\mathbb{R})$ denote the set of all $n \times n$ symmetric matrices and let $\mathrm{SkSym}_n(\mathbb{R})$ denote the set of all $n \times n$ skew symmetric matrices. Show that $\mathrm{Sym}_n(\mathbb{R})$ and $\mathrm{SkSym}_n(\mathbb{R})$ are subsaces of $M_n(\mathbb{R})$. Further show that

$$M_n(\mathbb{R}) = \mathrm{Sym}_n(\mathbb{R}) \oplus \mathrm{SkSym}_n(\mathbb{R}).$$

Exercise 2.15. Let V denote the subspace of $M_n(\mathbb{R})$ consisting of matrices (a_{ij}) such that $\sum_i a_{ii} = 0$. Show that V is a subspace of $M_n(\mathbb{R})$. Find a subspace W such that

$$M_n(\mathbb{R}) = V \oplus W.$$