LINEAR ALGEBRA- LECTURE 19

1. Matrix of a linear transformation

Having defined the matrix of a linear transformation let us look at some more properties and applications. We first recall that any $m \times n$ matrix A with entries in a field F gives rise to a linear transformation

$$A : F^n \longrightarrow F^m, \quad A(X) = AX$$

(which is left multiplication by the matrix A and is) usually denoted by A. We also know that given any linear transformation

$$T : F^n \longrightarrow F^m$$

there is a unique $m \times n$ matrix A such that T equals left multiplication by A. This matrix A is the matrix of T relative to the standard basis of F^n and F^m. In particular every linear transformation $T : F^n \longrightarrow F^m$ is actually left multiplication by a matrix.

We further note the following. Suppose that $T : V \longrightarrow W$ is a linear map between finite dimensional vector spaces. Let B, C be bases of V and W respectively. Let A be the matrix of T relative to the bases B and C. Then we have a diagram of vector spaces and linear maps as below.

$$\begin{tikzcd} V \arrow[r, "T"] & W \\ F^n \arrow[u, "B"] \arrow[r, "A"'] & F^m \arrow[u, "C"'] \end{tikzcd}$$

The linear transformations B, C are given by

$$B(X) = BX, \quad C(Y) = CY.$$

The linear maps B and C are both isomorphisms since B, C are bases of V and W respectively. We claim that we have equality

$$C \circ A = T \circ B \tag{1.0.1}$$

of composition of maps. To see this we let $B = (v_1, \ldots, v_n)$ and $C + (w_1, \ldots, w_m)$ be bases of V and W respectively. We now compute

$$TB(e_j) = T(v_j).$$

Assume that

$$T(v_j) = a_{1j}w_1 + \cdots + a_{mj}w_m$$

so that

$$C^{-1}(T(v_j)) = (a_{1j}, \ldots, a_{mj})^t$$

is the coordinate vector of $T(v_j)$ and also equals the j-th column vector of the matrix A. It is now clear that

$$A(e_j) = C^{-1}(T(v_j)) = C^{-1}TB(e_j)$$

thereby proving (1.0.1).

Remark 1.1. Since the linear transformation A is unique, whenever we have a diagram as above and (1.0.1) holds, the matrix A is always the matrix of T relative to the bases B and C. Note that we have the following observations

$$T(B) = CA, \quad AX = Y \tag{1.1.1}$$

where we recall that $T(B)$ is the ordered set

$$T(B) = (T(v_1), \ldots, T(v_n))$$

and the equation $AX = Y$ means that if $v \in V$ has coordinate vector X, then the coordinate vector Y of $T(v)$ can be computed by

$$AX = Y.$$

Example 1.2. Let A be a $m \times n$ matrix and consider the associated linear transformation $A : F^n \longrightarrow F^m$. Then the matrix of Arelative to the standard basis of F^n and F^m clearly equals the matrix A.

Example 1.3. Let $B = C$ be bases of a finite dimensional vector space V. Then the matrix of the identity map $\mathrm{id} : V \longrightarrow V$ relative to B and C equals the identity matrix.

Example 1.4. Let $V \xrightarrow{T} W \xrightarrow{S} U$ be linear transformations between finite dimensional vector spaces with bases B, C, D respectively. Let $m(T), m(S)$ denote the matrices of T, S respectively relative to the chosen bases. We then claim the the matrix $m(S \circ T)$ of $S \circ T$ relative to B and D equals the product

$$m(S \circ T) = m(S) \cdot m(T).$$

This follows from the diagram

$$\begin{tikzcd}
V \arrow[r, "T"] & W \arrow[r, "S"] & U \\
F^n \arrow[u, "B"] \arrow[r, "m(T)"'] & F^m \arrow[u, "C"'] \arrow[r, "m(S)"'] & F^r \arrow[u, "D"']
\end{tikzcd}$$

and the remark above.

Given a linear transformation $T : V \longrightarrow W$ between finite dimensional vector spaces, it is often useful to know if it is possible to chose bases so that the matrix of T has a nice form. we shall be interested in such results. Here is an example.

Proposition 1.5. Let $T : V \longrightarrow W$ be a linear transformation between finite dimensional vector spaces. Then there exists basis B of V and C of W such that the matrix $A = m(T)$ of T has the form

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \tag{1.5.1}$$

where I_r denotes the $r \times r$ identity matrix and r is the rank of T.

Proof. Is left as an exercise. $\square$

Definition 1.6. Given a $m \times n$ matrix A with entries in a field F, the span of the column vectors is a subspace W of F^m. The dimension of W is called the rank (or sometimes the column rank) of the matrix A.

The rank of a $m \times n$ matrix A is therefore the the rank of the linear transformation $A : F^n \longrightarrow F^m$. We note that if

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

then

$$\text{rank}(A) = \text{rank}(A^t).$$

The matrix form of the above proposition is the following.

Proposition 1.7. Let A be a $m \times n$ matrix. Then there exist invertible matrices P, Q such that the matrix A' given by

$$A' = Q^{-1}AP$$

has the form as in (1.5.1) and r is the rank of the matrix A.

Proof. Is left as an exercise. □

We now consider the question of how the matrix of a linear transformation changes if change the bases. More precisely, let $T : V \longrightarrow W$ be a linear transformation between finite dimensional vector spaces and $A = m(T)$ be the matrix of T relative to the bases B, C of V and W respectively. Suppose we are given basis B' of V and C' of W. Let $A' = m(T)$ relative to the bases B' and C'. Then we wish to understand how the matrices A and A' are related.

We know from our earlier observations that given the basis B of V every other basis B' is obtained as

$$B' = BP, \quad PX' = X$$

where P is an invertible matrix and X, X' denote the coordinate vectors in the bases B and B' respectively. Indeed, P is the basechange matrix from B to B'. Similarly we have an invertible matrix Q such that

$$C' = CQ, \quad QY' = Y.$$

Since A is the matrix of T, by (1.1.1) we also have that

$$AX = Y$$

and also

$$A'X' = Y'.$$

We may now compute

$$QY' = Y = AX = APX'$$

and hence

$$Y' = (Q^{-1}AP)X'. \tag{1.7.1}$$

But since A is the unique matrix such that (1.7.1) holds, we must have

$$A' = Q^{-1}AP.$$

Proposition 1.8. Let A be the matrix of a linear transformation $T : V \longrightarrow W$ relative to bases B, C of V, W respectively.

(1) The matrix A' of T relative to bases B', C' of V, W is of the form

$$A' = Q^{-1}AP$$

where P and Q are the basechange matrices from B to B' and C to C' respectively.

(2) The matrices A' that represent T with respect to other bases are of the form $A' = Q^{-1}AP$ where P, Q are invertible matrices.

Here is an example.

Example 1.9. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2z \end{pmatrix}.$$

The matrix of T in the standard bases is

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Let

$$B' = \left((1,2,0)^t, (0,1,0^t), (0,1,2)^t\right), \quad C' = \left((1,1)^t, (0,1)^t\right)$$

be two new bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Let P denote the basechange matrix from the standard basis of $\mathbb{R}^3$ to B' and Q denote the basechange matrix from the standard basis of $\mathbb{R}^2$ to C'. Then

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Thus the matrix of T relative to the bases B', C' is

$$A' = Q^{-1}AP.$$

Here is an application of the results that we have proved so far. Let A be a $m \times n$ matrix with entries in a field F. Then we have two linear transformations

$$A : F^n \longrightarrow F^m$$

We know, by Proposition 1.7, that there exist invertible matrices P, Q such that

$$A' = Q^{-1}AP$$

has the form in (1.5.1). Thus we obtain two diagrams

$$\begin{tikzcd} F^n \arrow[r, "A"] & F^m & F^m \arrow[r, "A^t"] \arrow[d, "Q^t"'] & F^n \arrow[d, "P^t"] \\ F^n \arrow[u, "P"] \arrow[r, "A'"'] & F^m \arrow[u, "Q"'] & F^m \arrow[r, "A'^t"'] & F^n \end{tikzcd}$$

We observe that as P, Q are isomorphisms, Q maps the column space of A' isomorphically onto the column space of A and hence

$$\operatorname{rank}(A) = \operatorname{rank}(A').$$

A similar argument shows that

$$\operatorname{rank}(A^t) = \operatorname{rank}(A'^t).$$

Thus we get the equality

$$\operatorname{rank}(A) = \operatorname{rank}(A^t).$$

Since the rank of A^t can be thought of as the dimension of the row vectors of A, we obtain the following.

Proposition 1.10. Let A be a $m \times n$ matrix with entries in a field F. Let $W \leq F^m$ and $U \leq F^n$ denote the vector spaces spanned by the column and the row vectors of A respectively. Then

$$\dim(W) = \dim(U).$$

The subspace U is called the row space of the matrix A and $\dim(U)$ the row rank of A. Thus the proposition expresses the equality of the row and the column rank of a matrix. Thus to compute the rank of a matrix it is enough to compute either the row rank or the column rank.

Here are some problems.

Exercise 1.11. Use Proposition 1.8 to show that Propositions 1.5 and 1.7 are equivalent.

Exercise 1.12. Let A be a $m \times n$ matrix. Show that $\mathrm{rank}(A) \leq \min\{m, n\}$.

Exercise 1.13. Let P be an invertible matrix. Show that $\mathrm{rank}(P^{-1}AP) = \mathrm{rank}(A)$.

Exercise 1.14. Let A be a $m \times n$ matrix. Show that the space of solutions has dimension at least $n - m$.

Exercise 1.15. Prove that every $m \times n$ matrix A has the form $A = XY^t$ where X, Y are m-dimensional column vectors.

Exercise 1.16. If A and B are row equivalent matrices, prove that $\mathrm{rank}(A) = \mathrm{rank}(B)$. Thus the rank of a matrix equals the rank of its row echelon form.

Exercise 1.17. Prove that $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$ and $\mathrm{rank}(A + B) \leq \mathrm{rank}(A) + \mathrm{rank}(B)$ whenever the product and sum are defined.

Exercise 1.18. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation whose kernel is a line through the origin. Can the image of T be a line through the origin?

Exercise 1.19. Write down a 3×3 matrix whose column space is a plane in $\mathbb{R}^3$. What is the row space in this case?

Exercise 1.20. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

Find invertible matrices P, Q such that $A' = Q^{-1}AP$ has the form in (1.5.1).