LINEAR ALGEBRA- LECTURE 3

1. Matrices - Row operations

Last time we noted that when we multiply an $n \times r$ matrix X on the left by an $n \times n$ matrix A to get a matrix Y, then each row vector Y_i of Y is a linear combination of the rows of X. Indeed we can write

$$Y_i = a_{i1}X_1 + a_{i2}X_2 + \cdots + a_{in}X_n.$$

We shall try to understand the product AX when the matrix A is of a special type.

Recall that e_{ij} is a matrix unit with entry 1 in the ij-th place and zero elsewhere. One defines three types of elementary matrices. These are square matrices and are defined as follows.

A type (i) elementary matrix A is a sum of the form

$$A = \mathbb{I}_n + ae_{ij}$$

where $i \neq j$, $a \neq 0$ and e_{ij} is the $n \times n$ matrix unit. Thus a type (i) elementary matrix has 1 on the diagonal and a nonzero entry a in the ij-th position and zero elsewhere. For example

$$\mathbb{I}_3 + 2e_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

is a 3×3 elementary matrix of type (i).

A type (ii) elementary matrix A is a sum of the form

$$A = \mathbb{I}_n - e_{ii} - e_{jj} + e_{ij} + e_{ji}.$$

where $i \neq j$. Thus a type (ii) elementary matrix is obtained from the identity matrix by replacing the i-th and the j-th diagonal entry by 0 and adding 1 to the ij-th and ji-th place. For example

$$\mathbb{I}_4 - e_{11} - e_{33} + e_{13} + e_{31} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a 4×4 elementary matrix of type (ii).

Finally, a type (iii) elementary matrix A is a sum of the form

$$A = \mathbb{I}_n - e_{ii} + ae_{ii}.$$

where $a \neq 0$. Thus a type (iii) elementary matrix is obtained from the identity matrix by replacing the i-th diagonal entry by a nonzero number.

We now wish to understand the product AX when A is one of the three types of elementary matrices above. So suppose $X = (x_{ij})$ is a $n \times r$ matrix and

$$AX = Y$$

and $A = \mathbb{I}_n + ae_{ij}$ is an elementary matrix of type (i) so that $i \neq j$ and $a \neq 0$. Thus

$$Y = AX = (\mathbb{I}_n + ae_{ij})X = X + (ae_{ij})X.$$

It is easy to convince oneself (see Lecture 2, Exercise 1.7) that if we form the product

$$ae_{ij}X = Y',$$

then

$$Y'_k = \begin{matrix} 0 & k \neq i \\ aX_j & k = i. \end{matrix}$$

Hence,

$$Y_k = \begin{matrix} X_k & k \neq i \\ X_i + aX_j & k = i. \end{matrix}$$

Thus left multiplication by an elementary matrix, $A = \mathbb{I}_n + ae_{ij}$, transforms a matrix X into a matrix Y that has the same rows as X except the i-th row which is now of the form

$$Y_i = X_i + aX_j.$$

The effect of left multiplication by elementary matrices of type (i)-(ii)-(iii) can be summed up as below. The verification is left as an exercise.

Proposition 1.1. Let X be an $n \times r$ matrix and A an elementray matrix with $AX = Y$.

(1) If $A = \mathbb{I}_n + ae_{ij}$ is an elementary matrix of type (i), then

$$Y_k = \begin{matrix} X_k & k \neq i \\ X_i + aX_j & k = i. \end{matrix}$$

Thus, left multiplication by an elementary matrix, $A = \mathbb{I}_n + ae_{ij}$, transforms a matrix X into a matrix Y that has the same rows as X except that $Y_i = X_i + aX_j$.

(2) If $A = \mathbb{I}_n - e_{ii} - e_{jj} + e_{ij} + e_{ji}$ is an elementary matrix of type (ii), then

$$Y_k = \begin{matrix} X_k & k \neq i, j \\ X_j & k = i \\ X_i & k = j. \end{matrix}$$

Thus, left multiplication by an elementary matrix $A = \mathbb{I}_n - e_{ii} - e_{jj} + e_{ij}$ transforms a matrix X into a matrix Y that has the same rows as X except that the i-th and the j-th rows get interchanged.

(3) If $A = \mathbb{I}_n - e_{ii} + ae_{ii}$ is an elementary matrix of type (ii), then

$$Y_k = \begin{matrix} X_k & k \neq i \\ aX_i & k = i. \end{matrix}$$

Thus left multiplication by an elementary matrix $A = \mathbb{I}_n - e_{ii} + ae_{ii}$ transforms a matrix X into a matrix Y that has the same rows as X except that the i-th row gets modified to aX_i.

An important fact about elementary matrices is that they are all invertible.

Lemma 1.2. Every elementary matrix is invertible.

Proof. The proof is by computation. We keep in mind the Exercise 1.8 (Lecture 2). Suppose $A = \mathbb{I}_n + ae_{ij}$ is an elementary matrix of type (i). Then $a \neq 0$ and $i \neq j$ and hence we obtain

$$(\mathbb{I}_n + ae_{ij})(\mathbb{I}_n - ae_{ij}) = \mathbb{I}_n + ae_{ij} - ae_{ij} - a^2 e_{ij} e_{ij} = \mathbb{I}_n$$

and

$$(\mathbb{I}_n - ae_{ij})(\mathbb{I}_n + ae_{ij}) = \mathbb{I}_n.$$

Thus $A = \mathbb{I}_n + ae_{ij}$ is invertible with inverse equal to $\mathbb{I}_n - ae_{ij}$.

Next suppose that

$$A = \mathbb{I}_n - e_{ii} - e_{jj} + e_{ij} + e_{ji}$$

is an elementary matrix of type (ii) so that $i \neq j$. Then $A^2 = \mathbb{I}_n$ so that A is its own inverse.

Finally, if

$$A = \mathbb{I}_n - e_{ii} + ae_{ii}$$

is an elementary matrix of type (iii) with $a \neq 0$, then

$$A^{-1} = \mathbb{I}_n - e_{ii} + \frac{1}{a}e_{ii}.$$

The final two checkings are left as an exercise. □

Keeping in mind the above proposition there are three type of operations, called row operations, that one can perform on a matrix. This is a process of transforming a matrix to another matrix which is hopefully simpler. The three types of row operations are as follows.

Let $X = (x_{ij})$ be a $n \times r$ matrix.

- (type (i) row operation) Replace the i-the row X_i of X by $X_i + aX_j$, $i \neq j$ and $a \neq 0$, keeping all other rows unchanged. This type of operation transforms a matrix X into another as below

$$\begin{pmatrix} - & X_1 & - \\ \vdots & \vdots & \vdots \\ - & X_i & - \\ \vdots & \vdots & \vdots \\ - & X_j & - \\ \vdots & \vdots & \vdots \\ - & X_n & - \end{pmatrix} \longrightarrow \begin{pmatrix} - & X_1 & - \\ \vdots & \vdots & \vdots \\ - & X_i + aX_j & - \\ \vdots & \vdots & \vdots \\ - & X_j & - \\ \vdots & \vdots & \vdots \\ - & X_n & - \end{pmatrix}$$

  It is clear that the resultant matrix by this row operation can be got by left multiplying X by an elementary matrix of type (i).
- (type (ii) row operation) Interchange the i-th and the j-th rows of X. This type of operation transforms the matrix X into another as below

$$\begin{pmatrix} - & X_1 & - \\ \vdots & \vdots & \vdots \\ - & X_i & - \\ \vdots & \vdots & \vdots \\ - & X_j & - \\ \vdots & \vdots & \vdots \\ - & X_n & - \end{pmatrix} \longrightarrow \begin{pmatrix} - & X_1 & - \\ \vdots & \vdots & \vdots \\ - & X_j & - \\ \vdots & \vdots & \vdots \\ - & X_i & - \\ \vdots & \vdots & \vdots \\ - & X_n & - \end{pmatrix}$$

  Again, the resultant matrix can be got by left multiplying the matrix X by an elementary matrix of type (ii).

- (type (ii) row operation) Multiply the i-th row X_i of X by a nonzero number. This type of operation transforms the matrix X into another as below
$$\begin{pmatrix} - & X_1 & - \\ \vdots & \vdots & \vdots \\ - & X_i & - \\ \vdots & \vdots & \vdots \\ - & X_n & - \end{pmatrix} \longrightarrow \begin{pmatrix} - & X_1 & - \\ \vdots & \vdots & \vdots \\ - & aX_i & - \\ \vdots & \vdots & \vdots \\ - & X_n & - \end{pmatrix}$$
The resultant matrix can be got by left multiplying the matrix X by an elementary matrix of type (iii).

From now on the symbol E (along with subscripts) will denote an elementary matrix. Having defined row operations, we may iteratively operate on an $n \times r$ matrix X by a sequence of s-many, say, row operations to obtain a matrix X'
$$X \overset{\text{row operation}}{\longrightarrow} \cdot \overset{\text{row operation}}{\longrightarrow} \cdots \overset{\text{row operation}}{\longrightarrow} X'.$$
We know that each row operation is equivalent to a left multiplication by a suitable elementary matrix. Thus, if after the first step we obtain the matrix Y, then
$$Y = E_1 X$$
for some elementary matrix E_1 and iterating this process we have
$$X' = E_s E_{s-1} \cdots E_1 X$$
for some elementary elementary matrices $E_1, \ldots, E_s$. We say that the matrix X' is obtained from X by row reduction. Two matrices A and B are said to be row-equivalent if B can be obtained from A by a sequence of row operations. Notice that this means that A can be obtained from B by a sequence of row operations too.

Here is an example. Let us apply row operations on the matrix
$$X = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{pmatrix}$$
as follows
$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{pmatrix} \overset{X_2:X_2+X_3}{\longrightarrow} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & 4 & 6 \\ 1 & -2 & 1 & 1 \end{pmatrix}$$
The notation $X_2 : X_2 + X_3$ means replace the second row X_2 by $X_2 + X_3$ a row operation of type (i). With this understanding we write down the steps that are self explanatory
$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{pmatrix} \overset{X_2:X_2+X_3}{\longrightarrow} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & 4 & 6 \\ 1 & -2 & 1 & 1 \end{pmatrix} \overset{X_3:X_3-X_1}{\longrightarrow} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & 4 & 6 \\ 0 & -4 & 0 & 1 \end{pmatrix} \overset{X_1:X_1+X_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 5 & 6 \\ 0 & -2 & 4 & 6 \\ 0 & -4 & 0 & 1 \end{pmatrix}$$
$$\overset{X_2:(-1/2)X_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 5 & 6 \\ 0 & 1 & -2 & -3 \\ 0 & -4 & 0 & 1 \end{pmatrix} \overset{X_4:X_4+4X_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 5 & 6 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -8 & -11 \end{pmatrix} \overset{X_3:(-1/8)X_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 5 & 6 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 11/8 \end{pmatrix}$$
$$\overset{X_1:X_1-5X_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & -7/8 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 11/8 \end{pmatrix} \overset{X_2:X_2+2X_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & -7/8 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 11/8 \end{pmatrix}$$

It is possible to use row reduction of matrices to solve systems of linear equations. Recall that given a system of m linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1.2.1}$$

in n variables, we may denote this system by the single equation

$$AX = B$$

where $A = (a_{ij})$ is the matrix of coefficients and X and B are the column vectors

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

We then consider the matrix

$$M = [A|B] = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{mn} & b_m \end{array}\right)$$

The $m \times (n+1)$ matrix M is called the augmented matrix associated to the above system of linear equations. If we now perform a single row operation on the matrix M, then we get a matrix M'

$$M' = \left(\begin{array}{ccc|c} a'_{11} & \cdots & a'_{1n} & b'_1 \\ \vdots & & \vdots & \vdots \\ a'_{n1} & \cdots & a'_{mn} & b'_m \end{array}\right)$$

This is the augmented matrix associated to the system of equations

$$A'X = B'.$$

We note the following fact.

Proposition 1.3. The systems of equations $AX = B$ and $A'X = B'$ have the same solutions.

Proof. This follows from Proposition 2.4 (Lecture 1). □

Here are some examples.

Example 1.4. Consider the following system of 3 linear equations in 3 variables

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ -x_1 + 0x_2 + 3x_3 &= 5 \\ x_1 - 2x_2 + x_3 &= 1 \end{aligned}$$

The augmented matrix corresponding to this system is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{array}\right) \tag{1.4.1}$$

This augmented matrix can be row reduced to the matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -7/8 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 11/8 \end{array}\right)$$

This is the augmented matrrix xorresponding to the system of equations

$$\begin{aligned} x_1 + 0x_2 + 0x_3 &= -7/8 \\ 0x_1 + x_2 + 0x_3 &= -1/4 \\ 0x_1 + 0x_2 + x_3 &= 11/8 \end{aligned} \tag{1.4.2}$$

By Proposition 1.3 the systems (1.4.1) and (1.4.2) have the same solutions so we are done.

Exercise 1.5. Consider the following system of 3 linear equations in 2 variables.

$$\begin{aligned} -x_1 + ix_2 &= 0 \\ -ix_2 + 3x_2 &= 0 \\ x_1 + 2x_2 &= 0 \end{aligned}$$

This is a homogeneous system and therefore we need not consider the augmented matrix of the system. We just try to row reduce the matrix of coefficients.

$$\begin{pmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{pmatrix} \xrightarrow{X_1:X_1+X_2} \begin{pmatrix} 0 & 2+i \\ -i & 3 \\ 1 & 2 \end{pmatrix} \xrightarrow{X_2:X_2+iX_3} \begin{pmatrix} 0 & 2+i \\ 0 & 3+2i \\ 1 & 2 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Thus the given system of equations is equivalent to the system of equations

$$\begin{aligned} 0x_1 + x_2 &= 0 \\ x_1 + 0x_2 &= 0. \end{aligned}$$

Hence the the given system of equations has only the trivial solution $x_1 = x_2 = 0$.

Here are some exercises.

Exercise 1.6. Complete the proof of Proposition 1.1

Exercise 1.7. Complete the proof of Lemma 1.2.

Exercise 1.8. Find all solutions to the system of equations

$$\begin{aligned} (1-i)x_1 - ix_2 &= 0 \\ 2x_1 + (1-i)x_2 &= 0. \end{aligned}$$

Exercise 1.9. If

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix}$$

find all solutions to the homogeneous system $AX = 0$.

Exercise 1.10. If

$$A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

find all solutions of the system $AX = 2X$ and $AX = 3X$. Here X is a 3×1 column vector.

Exercise 1.11. Give examples of matrices A, B that are not row-equivalent.

Exercise 1.12. Find all solutions to the system of equations $AX = B$ when

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & 2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$