LINEAR ALGEBRA- LECTURE 4

1. Matrices - Row operations...continued...

Our discussion in the previous sections shows that matrices are very well suited to understanding a method of finding solutions to a system of linear equations

$$AX = B.$$

We use row operations on the augmented matrix $(A|B)$ to get a possibly simpler augmented matrix $(A'|B')$ that represents the system of equations

$$A'X = B'$$

which has the same set of solutions as the system $AX = B$.

Our interest now is to understand the procedure of row reduction a bit more closely. In particular, we will try to understand if there is a particularly nice form to which every matrix can be row reduced. Recall our convention : for a matrix A, A_i denotes the i-th row vector of A. We make the following definition.

Definition 1.1. A matrix A is said to be a row echelon matrix (or to be in row echelon form) if the following conditions are satisfied.

(1) The first non-zero entry in each row is 1. This is called a pivot.
(2) The first non-zero entry of the $(i+1)$-th row is to the right of the first non-zero entry of the i-th row. That is, the pivot in the $(i+1)$-th row is to the right of the pivot in the i-th row.
(3) The entries above a pivot are zero.

Observe that in a row echelon matrix A if the i-th row A_i consists of zeros, then by property (2) each row A_j, $j > i$ also consists of zeros. Also note that all entries (other than the pivot) in a column containing a pivot are zero. This follows from properties (3) and (2).

Just so that we understand the definition here is an example. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

This is not a row echelon matrix. This violates all the conditions.

It turns out that every matrix can be row reduced to a row echelon matrix. In other words, every matrix can be converted to a row echelon matrix by a sequence of elementary row operations. We shall prove this next.

Here are some problems.

Exercise 1.2. Convert the following matrices to a matrix in the row echelon form.

$$\begin{pmatrix} 2 & 3 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 2 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 & 0 \\ 3 & 0 & -3 & 1 \\ 5 & 4 & 2 & 1 \end{pmatrix}$$

Exercise 1.3. Find all solutions of the equation $x_1 + x_2 + 2x_3 - x_4 = 3$.