LINEAR ALGEBRA- LECTURE 6

1. Row operations....final part

We recall that we have seen several equivalent ways of showing that a matrix is invertible. Indeed, we had proved the following proposition.

Proposition 1.1. Let A be a square matrix. Then the following statements are equivalent.

(1) A is row equivalent to the identity matrix.
(2) A is a product of elementary matrices.
(3) A is invertible.
(4) The system of homogeneous equations $AX = 0$ has only the trivial solution.

We had seen some consequences of the above proposition. We record one more.

Proposition 1.2. Let A, B be square matrices such that $AB = \mathbb{I}$ (or $BA = \mathbb{I}$), then A is invertible and $A^{-1} = B$.

Proof. We first row reduce A to a row echelon matrix A'. Then either $A' = \mathbb{I}$ or the bottom row of A' is zero. Now we may write

$$E_s E_{s-1} \cdots E_1 A = A'$$

where E_i are elementary matrices, so that

$$A'B = E_s E_{s-1} \cdots E_1.$$

This implies that the bottom row of A' is not zero and hence $A' = \mathbb{I}$. This implies, by the above proposition, that A is invertible and

$$A^{-1} = E_s E_{s-1} \cdots E_1 = B.$$

□

Remark 1.3. We remark that the row echelon form of a matrix is essentially unique. In other words suppose that A_1 and A_1 are two row echelon matrices obtained from the same matrix A by possiby different sequence of elementary row operations. THen $A_1 = A_2$

Here are two examples that make use of Proposition 1.1. The proof of the second example was pointed out by a friend in the class.

Example 1.4. Let A be a square matrix. Then A is invertible if and only if the row reduced echelon form of A does not contain a row of zeros. This is essentially the equivalence of (i) and (iii) in Proposition,1.1.

Example 1.5. Let A, B square matrices such that AB is invertible. Then both A and B are invertible. To see this we first row reduce AB to the identity matrix by wrting

$$E_s \cdots E_1(AB) = \mathbb{I}$$

where E_i are elementary matrices. Thus

$$B^{-1} = E_s \cdots E_1 A$$

and hence B is invertible by Proposition 1.2. Consequently, A is also invertible.

Another operation that one can perform on a matrix is the transpose. Here is the formal definition.

Definition 1.6. Given a matrix $A = (a_{ij})$ the matrix $B = (b_{ij})$ where

$$b_{ij} = a_{ji}$$

is called the transpose of A and is usually denoted by A^t.

The way in which the transpose of a matrix is connected with the alegbraic operations on matrices is noted below. The proof is left as an exercose.

Lemma 1.7. Given matrices A, B and a scalar r we have

(1) $(A + B)^t = A^t + B^t$.
(2) $(AB)^t = B^t A^t$.
(3) $(rA)^t = rA^t$.
(4) $(A^t)^t = A$.

Proof. Exercise. □

Here are some problems.

Exercise 1.8. Suppose that two square matrices A, B are row equivalent. Is it true that A is invertible if and only if B is invertible?

Exercise 1.9. Let A, B be two $n \times n$ matrices. Prove that if

$$AX = BX$$

for every column vector X, then $A = B$.

Exercise 1.10. A matrix A is said to be symmetric if $A = A^t$. Prove that for any matrux A, the matrix AA^t is symmetric and that if A is square matrix then $A + A^t$ is symmetric.

Exercise 1.11. Let A, B be two symmetric square matrices. Prove that AB is symmetric if and only if $AB = BA$.

Exercise 1.12. Prove that the inverse of an invertible symmetric matrix is also symmetric.

2. Determinants

In this section we shall discuss the definition of the determinant of a square matrix and understand some of its properties. Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries. The determinant is a function denoted by det

$$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}.$$

Our aim is to define this function and study some of its properties.

The definition of det is arrived at recursively in the following fashion. The determinant of a 1×1 matrix $A = (a)$ is defined to be

$$\det(A) = a. \tag{2.0.1}$$

If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we set

$$\det(A) = ad - bc. \tag{2.0.2}$$

More generally we consider a $n \times n$ matrix $A = (a_{ij})$ and assume that the determinant is defined for matrices of size less than n. We first define the ij-minor A_{ij} of the matrix A. This is defined as the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i-th row and the j-th column. For example if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

then

$$A_{12} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}.$$

is obtained from A by deleting the first row and the second column.

Definition 2.1. Let $A = (a_{ij})$ be a $n \times n$ matrix with $n \geq 2$. Then we define

$$\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + \cdots \pm a_{n1}\det A_{n1}. \tag{2.1.1}$$

This along with (2.0.1) defines the determinant function recursively for all n.

It is clear that if A is a 2×2 matrix, then the definition of $\det(A)$ in (2.1.1) reduces to the one expressed in (2.0.2). The definition in (2.1.1) is referred to as the expansion by minors on the first column. Here is an example. If A is the 3×3 matrix above, then

$$\det(A) = 1 \cdot \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 4 \cdot \det \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} + 7 \cdot \det \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$$

The fundamental properties that the determinant function satisfies is expressed in the theorem below. We shall soon show that the detereminant function is the only function that satisfies the three conditions in the theorem below.

Theorem 2.2. The determinant function $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ satisfies the following conditions.

(1) $\det(\mathbb{I}_n) = 1$.
(2) det is linear in the rows of A.
(3) If two adjacent rows of A are equal, then $\det(A) = 0$.

Before we start the proof of the theorem let us try to understand the meaning of the statement (2) of the theorem. Suppose $A = (a_{ij})$ is written in the form

$$A = \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i & - \\ & \vdots & \\ - & A_n & - \end{pmatrix}$$

Now suppose that the i-th row vector A_i splits as $A_i = A_i' + A_i''$. Then the condition (2) in theorem says two things :

$$\det \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i' + A_i'' & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} = \det \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i' & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} + \det \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i'' & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} \tag{2.2.1}$$

and

$$\det\begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & r\cdot A_i & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} = r\cdot\det\begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} \tag{2.2.2}$$

where r is a scalar. For example

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 1 & 1 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 \\ 2+1 & 1+3 & 4+1 \\ 1 & 1 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We now prove the theorem.

Proof. The proof of each case is by a suitable induction hypothesis. It can be easily verified that in each case the induction can be started.

The proof of (1) is an easy exercise in applying the definition of the determinant and using induction. This verification is left as an exercise.

We shall now prove (2). We wish to show therefore that

$$\det\begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i' + A_i'' & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} = \det\begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i' & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} + \det\begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i'' & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} \tag{2.2.3}$$

Let A' and A'' denote the two matrices on the right hand side. Our induction hypothesis is that the determinant is linear in the rows of matrices of size $(n-1)\times(n-1)$. We now compute

$$\det(A') = a_{11}\det(A'_{11}) - a_{21}\det(A'_{21}) + \cdots \pm a'_{i1}\det(A'_{i1}) \pm \cdots \pm a_{n1}\det(A'_{n1}) \tag{2.2.4}$$

and

$$\det(A'') = a_{11}\det(A''_{11}) - a_{21}\det(A''_{21}) + \cdots \pm a''_{i1}\det(A''_{i1}) \pm \cdots \pm a_{n1}\det(A''_{n1}) \tag{2.2.5}$$

We now observe the following facts. If $k \neq i$, then by our induction hypothesis

$$\det(A'_{k1}) + \det(A''_{k1}) = \det(A_{k1})$$

and

$$\pm a'_{i1}\det(A'_{i1}) \pm a''_{i1}\det(A''_{i1}) = \pm(a'_{i1} + a''_{i1})\det(A_{i1})$$

since

$$A'_{i1} = A''_{i1} = A_{i1}.$$

This shows that

$$\det(A') + \det(A'') = \det(A)$$

and completes the proof of (2.2.3). That (2.2.2) holds is left as an exercise. This completes the proof of (2). We have assumed that

$$A_i' = (a'_{i1}, \ldots, a'_{in}), \quad A_i'' = (a''_{i1} \ldots, a''_{in}).$$

Finally, we turn to the proof of (3). We assume that $A_i = A_{i+1}$ and wish to show that $\det(A) = 0$. Now

$$\det(A) = a_{11}\det(A_{11}) - \cdots\cdots \pm a_{n1}\det(A_{n1}). \tag{2.2.6}$$

we now observe that if $k \neq i, i+1$, then the minor A_{k1} has two equal rwos and hence, by induction hypothesis, $\det(A_{k1}) = 0$ for $k \neq i, i+1$. Thus (2.2.6) reduces to

$$\det(A) = \pm a_{i1}\det(A_{i1}) \pm a_{i+1,1}\det(A_{i+1,1})$$

where the two terms on the right have opposite signs. Since

$$a_{i1} = a_{i+1,1}$$

and

$$A_{i1} = A_{i+1,1}$$

we conclude that $\det(A) = 0$. This completes the proof of the theorem. $\square$

Here are some problems.

Exercise 2.3. Compute the determinant of the $n \times n$ matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{matrix} 0 & \text{if } i + j \neq n + 1 \\ 1 & \text{otherwise} \end{matrix}$$

Exercise 2.4. Evaluate the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & \cdots & n \\ 3 & 3 & 3 & \cdots & n \\ \vdots & & & & \vdots \\ n & n & n & n\cdots & n \end{pmatrix}$$

Exercise 2.5. For a $n \times n$ matrix A, what is $\det(-A)$?