LINEAR ALGEBRA- LECTURE 7

1. Determinants...Further proeprties

Recall that if $A = (a_{ij})$ be a $n \times n$ matrix with $n \geq 2$, then the determinant of A is defined by

$$\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + \cdots \pm a_{n1}\det A_{n1}. \tag{1.0.1}$$

This defines the determinant function recursively for all n where the determinant of a 1×1 matrix is defined in an obvious manner.

Recall that we had also seen the proof of the following theorem which states the three fundamental properies of the determinant function.

Theorem 1.1. The determinant function $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ satisfies the following conditions.

(1) $\det(\mathbb{I}_n) = 1$.
(2) det is linear in the rows of A.
(3) If two adjacent rows of A are equal, then $\det(A) = 0$.

We shall now investigate some further properties of the determinant function. We emphasize that the properties that we shall derive depend solely on the fact that the determinant function satisfies the three conditions of the above theorem and not on the defining formula of the determinant.

Lemma 1.2. Suppose A is $n \times n$ matrix. Then the determinant is unchanged when the i-th row A_i is changed to $A_i + rA_{i+1}$.

Proof. As mentioned earlier we will derive this just from the fact that the determinant satisfies the three properties of the above theorem. By the linearity of the determinant we have

$$\det \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i + rA_{i+1} & - \\ - & A_{i+1} & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} = \det \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i & - \\ - & A_{i+1} & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} + r \cdot \det \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_{i+1} & - \\ - & A_{i+1} & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} \tag{1.2.1}$$

The last determinant is zero by property (3) of the above theorem. This completes the proof. □

Lemma 1.3. If two adjacent rows of a matrix are interchanged, then the determinant is multiplied by -1.

Proof. We look at the following sequence of equalities of determinant of matrices

$$\det \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i & - \\ - & A_{i+1} & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} = \det \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i & - \\ - & A_{i+1} - A_i & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} = \det \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i + (A_{i+1} - A_i) & - \\ - & A_{i+1} - A_i & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} \tag{1.3.1}$$

where the equalities are due to Lemma 1.2. Next we see that

$$\det \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i & - \\ - & A_{i+1} & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} = \det \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_{i+1} & - \\ - & A_{i+1} & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} + \det \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_{i+1} & - \\ - & -A_i & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} = -\det \begin{pmatrix} - & A_1 & - \\ & \vdots & \\ - & A_{i+1} & - \\ - & A_i & - \\ & \vdots & \\ - & A_n & - \end{pmatrix} \tag{1.3.2}$$

where the equalities follow from properties (2) and (3) of the above theorem. This completes the proof. □

The property (3) in Theorem 1.1 says that if two adjacent rows of a matrix A are equal, then $\det(A) = 0$. This remains true even if the rows are not adjacent.

Lemma 1.4. If two rows of a matrix A are equal, then $\det(A) = 0$.

Proof. We keep interchanging the rows of A to get a matrix A' in which now two rows are adjacent. Then

$$\det(A) = \pm\det(A') = 0.$$

The first equality follows from Lemma 1.3 and the second equality follows from Theorem 1.1 (3). □

Lemmas 1.2 and 1.3 also remain true more generally. We note this below. The proofs in both cases are left as exercises.

Lemma 1.5. If a multiple of a row is added to another row, then the determinant remains unchanged.

Proof. Exercise. □

Lemma 1.6. If two rows of a matrix are interchanged, then the determinant changes sign.

Proof. Exercise. □

The linearity proprty of the determinant shows, as can be easily checked, that if a matrix A has $A_i = 0$ then $\det = 0$. Let us look at the last two lemmas a bit more closely.

Lemma 1.5 says that the determinant of a matrix A remains unchanged for example when the i-th row A_i is changed to $A_i + rA_j$. In particular the $\det(A)$ does not change after performing an elementary row operation of type (i) on A. In other words, if E an elementary matrix of type (i), then

$$\det(EA) = \det(A) \tag{1.6.1}$$

Similarly, we may conclude from Lemma 1.6 that if E is an elementary matrix of typre (ii), then

$$\det(EA) = -\det(A). \tag{1.6.2}$$

Finally, if $E = \mathbb{I} - e_{ii} + rre_{ii}$ is an elementary matrix of type (iii), then

$$\det(EA) = r \cdot \det(A). \tag{1.6.3}$$

This is a consequence of the linearity of the determinant. These observations allow us to compute the determinant of the elementary matrices.

Lemma 1.7. For an elementary matrix E we have

$$\det(E) = \begin{array}{ll} 1 & \text{if } E \text{ is of type (i)} \\ -1 & \text{if } E \text{ is of type (ii)} \\ r & \text{if } E = I - e_{ii} + re_{ii} \text{ is of type (iii)} \end{array}$$

Proof. Take $A = \mathbb{I}$ in (1.6.1), (1.6.2) and (1.6.3). □

Lemma 1.8. Let E be an elementary matrix. Then for any matrix A we have

$$\det(EA) = \det(E) \cdot \det(A). \tag{1.8.1}$$

Proof. This follows from the equations (1.6.1), (1.6.2), (1.6.3) and Lemma 1.7. □

At this point we again emphasize that all the statements made after Theorem 1.1 have ony used the fact that the determinant satisfies the three properties in Theorem 1.1. We now observe the very interesting fact that we can now compute the determinant of a square matrix A as follows.

Suppose A is a square matrix. We then row reduce A to a row echelon matrix A'. The matrix A' is either the identity matrix (when A is invertible) or has the bottom row to be zero (when A is not invertible). Now we may write

$$A' = E_s \cdots E_1 A$$

and therefore

$$\det(A') = \det(E_s) \cdots \det(E_1) \det(A). \tag{1.8.2}$$

The equation (1.8.2) gives us a formula to compute $\det(A)$. At this point we note the uniqueness of the determinant function.

Theorem 1.9. Let $f : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ be a function that satisfies the conditions (1)-(2)-(3) of Theorem 1.1. Then $f(A) = \det(A)$ for every matrix $A \in M_n(\mathbb{R})$.

Proof. A moments thought will tell us that Lemmas 1.2 to 1.8 remain valid with det replaced by f and so do all the equations. Thus the final equation (1.8.2) tells us that $f(A) = \det(A)$. □

Observe that it follows from equation (1.8.2) that a matrix A is invertible if and only if

$$\det(A) \neq 0. \tag{1.9.1}$$

We now state a very important property of the determinant.

Theorem 1.10. Let $A, B \in M_n(\mathbb{R})$. Then

$$\det(AB) = \det(A) \cdot \det(B). \tag{1.10.1}$$

Proof. Suppose that A is not invertible. Then, by (1.9.1), the right hand side of (1.10.1) is zero. Since A is not invertible the product AB is not invertible. Hence the left hand side of (1.10.1) is also zero.

Next assume that A is invertible. Then A is a product of elementary matrices, say,

$$A = E_s \cdots E_1$$

and hence we have

$$\det(AB) = \det(E_s \cdots E_1\, B) = \det(E_s \cdots E_1)\det(B) = \det(A)\det(B).$$

This completes the proof. □

Theorem 1.11. For a square matrix A, we have $\det(A) = \det(A^t)$.

Proof. This is an interesting exercise. □

Remark 1.12. The determinant function was defined as a function

$$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$$

and we did not use any properties of the real numbers in our discussion of the determinant and its properties. The definition of the determinant can be made as in (1.0.1) for matrices with complex entries and all the above results remain valid. Thus we also have the determinant function

$$\det : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$$

where $M_n(\mathbb{C})$ denotes the set of $n \times n$ matrices with complex entries and all the above results remain valid for the complex determinant.

Here are some problems.

Exercise 1.13. Complete the proofs of Lemma 1.5, 1.6

Exercise 1.14. Let a row of a square matrix A be zero. Show that $\det(A) = 0$. What happens if a column is zero?

Exercise 1.15. Show that a matrix A is invertible if and only if $\det(A) \neq 0$.

Exercise 1.16. Let E be an elementary matrix of type (i). Is the transpose E^t also an elementary matrix? If so what is the type of E^t. What can you say about $\det(E)$? Answer the same questions when E is an elementary matrix of types (ii),(iii).

Exercise 1.17. Prove Theorem 1.11.

Exercise 1.18. Convince yourselves that Theorem 1.1 (2)-(3), Lemma 1.2, Lemma 1.3 remain valid if the word "row" is changed to column in each statement.

Exercise 1.19. For square matrices A, B show that $\det(AB) = \det(BA)$.

Exercise 1.20. Given a square matrix $A = (a_{ij})$, the trace of A denoted by $\mathrm{tr}(A)$ is defined to be

$$\mathrm{tr}(A) = \sum_i a_{ii}.$$

Prove the following.

1. $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
2. If B is invertible, then $\mathrm{tr}(A) = \mathrm{tr}(BAB^{-1})$.

Exercise 1.21. Find a 2×2 matrix A such that $A^2 = -\mathbb{I}_2$.

Exercise 1.22. Find a representation of the complex numbers by real 2×2 matrices which is compatible with addition and multiplication.

Exercise 1.23. The expansion of $\det(\mathbb{I} + rA)$ gives a polynomial in r. What is the constant term and the coefficient of r in this expansion?

Exercise 1.24. If A is invertibe show that

$$\left(A^{-1}\right)^t = \left(A^t\right)^{-1}.$$

Exercise 1.25. Show that the area of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is equal to

$$\frac{1}{2} \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

Exercise 1.26. Compute the determinant of

$$\begin{pmatrix} 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & e & f \\ p & q & r & s & t \\ u & v & w & x & y \end{pmatrix}$$