LINEAR ALGEBRA- LECTURE 7

1. Determinants ....final part

Let us recall the final part of our discussion in the last discussion. We had checked that if E is an elementary matrix of whichever type and A is any matrix, then

$$\det(EA) = \det(E) \cdot \det(A).$$

It therefore, follows by induction, that if $E_1, \ldots, E_s$ are elementary matrices, then for any matrix A we have

$$\det(E_s \cdots E_1 A) = \det(E_s \cdots E_1) \cdot \det(A) = \det(E_s) \cdots \det(E_1) \cdot \det(A).$$

We can now prove the following.

Proposition 1.1. A matrix $A \in M_n(\mathbb{R})$ is invertible if and only of $\det(A) \neq 0$.

Proof. Row reduce A to a row echelon matrix A' and write

$$A' = E_s \cdots E_1 A$$

where E_i are elementary matrices. Then A is invertible if and only if $A' = I$. Thus A is invertible if and only if

$$\det(E_s) \cdots \det(E_1) \cdot \det(A) \neq 0.$$

Thus A is invertible if and only if $\det(A) \neq 0$. Note that $\det(E_i) \neq 0$. □

Theorem 1.2. Let $A, B \in M_n(\mathbb{R})$. Then

$$\det(AB) = \det(A) \cdot \det(B). \tag{1.2.1}$$

Proof. Suppose that A is not invertible. Then, by Proposition 1.1, the right hand side of (1.2.1) is zero. Since A is not invertible the product AB is not invertible. Hence the left hand side of (1.2.1) is also zero.

Next assume that A is invertible. Then A is a product of elementary matrices, say,

$$A = E_s \cdots E_1$$

and hence we have

$$\det(AB) = \det(E_s \cdots E_1\, B) = \det(E_s \cdots E_1) \det(B) = \det(A) \det(B).$$

This completes the proof. □

Theorem 1.3. For a square matrix A, we have $\det(A) = \det(A^t)$.

Proof. This is an interesting exercise. □

It now follows that most of the results that we proved earlier are now true if we replace "row" by "column" in the statements of the above results. We state them without proof.

Theorem 1.4. The determinant function has the following properties.

(1) the determinant is linear in the columns of a matrix.

(2) If two columns of a matrix A are equal, then $\det(A) = 0$.
(3) If a multiple of a column is added to another column, then the determinant is unchanged.
(4) If two columns are interchanged, then the determinant changes sign.

Remark 1.5. The determinant function was defined as a function

$$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$$

and we did not use any properties of the real numbers in our discussion of the determinant and its properties. The definition of the determinant can be made as in (1.0.1) for matrices with complex entries and all the above results remain valid. Thus we also have the determinant function

$$\det : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$$

where $M_n(\mathbb{C})$ denotes the set of $n \times n$ matrices with complex entries and all the above results remain valid for the complex determinant.

There are other definitions of the determinant of a matrix $A = (a_{ij})$ that one may use. For example we may choose to expand by minors along the j-th column. This gives us the following expression for the determinant

$$\begin{aligned} \det(A) &= (-1)^{1+j} a_{1j} \det(A_{1j}) + (-1)^{2+j} a_{2j} \det(A_{2j} + \cdots + (-1)^{n+j} a_{nj} \det(A_{nj}) \\ &= \textstyle\sum_k (-1)^{k+j} a_{kj} \det(A_{kj}). \end{aligned} \tag{1.5.1}$$

One could also expand by minors on the rows. For example, expaanding by minors on the i-th row yields the following expresseion for the determinant of the matrix $A = (a_{ij})$

$$\begin{aligned} \det(A) &= (-1)^{1+i} a_{i1} \det(A_{i1}) + (-1)^{2+i} a_{i2} \det(A_{i2} + \cdots + (-1)^{n+i} a_{in} \det(A_{in}) \\ &= \textstyle\sum_k (-1)^{k+i} a_{ik} \det(A_{ik}). \end{aligned} \tag{1.5.2}$$

That the two expressions (1.5.1) and (1.5.2) yield the same value of the determinant is a consequence of the fact both these expressions satisfy the three properties that are funndamental to out definition of the determinant and therefore by uniqueness the above values must agree with the usual determinant.

We shall see one very useful method of computing the inverse of a matrix. This involves introducing a new matrix called the cofactor matrix. This is defined as follows. Given a matrix $A = (A_{ij})$ define

$$c_{ij} = (-1)^{i+j} \det(A_{ij}).$$

Let C denote the matric $C = (c_{ij})$.

Definition 1.6. Given a matrix $A = (a_{ij})$, let $C = (c_{ij})$ be the matrix defined above. Then the cofactor matrix of A, denoted by $\mathrm{cof}(A)$ is defined to be

$$\mathrm{cof}(A) = C^t$$

the transpose of the matrix C.

For example, let A be the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

We then compute

$$\begin{array}{lll} c_{11} = 1, & c_{12} = -6, & c_{13} = 3 \\ c_{21} = -3, & c_{22} = 2, & c_{23} = -1 \\ c_{31} = 1, & c_{32} = 2, & c_{33} = -5 \end{array}$$

Thus the matrix C equals

$$C = \begin{pmatrix} 1 & -6 & 3 \\ -3 & 2 & -1 \\ 1 & 2 & -5 \end{pmatrix}$$

and the cofactor matrix of A is

$$\mathrm{cof}(A) = C^t = \begin{pmatrix} 1 & -3 & 1 \\ -6 & 2 & 2 \\ 3 & -1 & -5 \end{pmatrix}$$

The cofactor matrix can be used to compute the inverse of an invertible matrix. This is contained in the following theorem.

Theorem 1.7. Let $A = (a_{ij})$ be a square matrix and $B = \mathrm{cof}(A)$ be the cofactor matrix of A. Then

$$BA = AB = \alpha I$$

where $\alpha = \det(A)$. Thus if $\alpha \neq 0$, then $A^{-1} = (1/\alpha)B$.

Proof. This is a simple verification and is left as an exercise. □

Here are some problems.

Exercise 1.8. Let a row of a square matrix A be zero. Show that $\det(A) = 0$. What happens if a column is zero?

Exercise 1.9. Let E be an elementary matrix of type (i). Is the transpose E^t also an elementary matrix? If so what is the type of E^t. What can you say about $\det(E)$? Answer the same questions when E is an elementary matrix of types (ii),(iii).

Exercise 1.10. Prove Theorem 1.3.

Exercise 1.11. Prove Theorem 1.7.

Exercise 1.12. For square matrices A, B show that $\det(AB) = \det(BA)$.

Exercise 1.13. Given a square matrix $A = (a_{ij})$, the trace of A denoted by $\mathrm{tr}(A)$ is defined to be

$$\mathrm{tr}(A) = \sum_i a_{ii}.$$

Prove the following.

(1) $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
(2) If B is invertible, then $\mathrm{tr}(A) = \mathrm{tr}(BAB^{-1})$.

Exercise 1.14. Find a 2×2 matrix A such that $A^2 = -\mathbb{I}_2$.

Exercise 1.15. Find a representation of the complex numbers by real 2×2 matrices which is compatible with addition and multiplication.

Exercise 1.16. The expansion of $\det(\mathbb{I} + rA)$ gives a polynomial in r. What is the constant term and the coefficient of r in this expansion?

Exercise 1.17. If A is invertibe show that

$$\left(A^{-1}\right)^t = \left(A^t\right)^{-1}.$$

Exercise 1.18. Show that the area of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is equal to

$$\frac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

Exercise 1.19. Compute the determinant of

$$\begin{pmatrix} 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & e & f \\ p & q & r & s & t \\ u & v & w & x & y \end{pmatrix}$$