LINEAR ALGEBRA- PROBLEMS SET

1. Exercises

(1) For a set X, let $P(X)$ denote the set of all subsets of X. Show that $P(X)$ is a vector space over the field F_2 with two elements. What is its dimension? Write down a basis when it is finite dimensional.

(2) Let $X = \{1, 2, 3, 4, 5\}$. Describe the subspace of $P(X)$ (Problem 1) spanned by the set of vectors $S = (\{1, 2, 3\}, \{2, 3, 4\}, \{1, 5\})$.

(3) Consider the vectors $v = (1, 3, 2)^t, w = (-2, 4, 3)^t$ in $\mathbb{R}^3$. show that the span of the set $S = (v, w)$ equals

$$\text{span}(S) = \{(x_1, x_2, x) \in \mathbb{R}^3 \,:\, x_1 - 7x_2 + 10x_3 = 0\}.$$

(4) Let A, B be subsets of a vector space V. Show that

$$\text{span}(A) \cup \text{span}(B) \subseteq \text{span}(A \cup B)$$
$$\text{span}(A) \cap \text{span}(B) \subseteq \text{span}(A \cap B)$$

and give examples to show that the inclusions may be proper.

(5) Let W be a subspace of a vector space V. Let $v, u \in V$ be two vectors such that $v + u \in W$. show that either both v and u beong to W or neither belong to W. If a is a scalar such that $av \in W$ show that $v \in W$.

(6) Let S be a subset of a vector space V. Show that span(S) is the intersection of all subspaces of V that contain S.

(7) Let U, W be subspaces of V. Show that $U \cup W$ is a subspace of V if and only if either $U \subseteq W$ or $W \subseteq U$.

(8) Let (u, v) be a linearly independent subset of a vector space V. Show that $(u + av, u + bv)$ is linearly independent whenever $a \neq b$.

(9) Let (u, v, w) be linearly independent set in V. when is the set $(u + v, v + w, u + w)$ also linearly independent?

(10) Let S be a linearly independent subset of a subspace W of a vector space V. Let $S' \subseteq (V - S)$ be a linearly independent set. Does it follow that $S \cup S'$ is linearly independent?

(11) Let $V \subseteq \mathbb{R}^4$ be the subset defined by

$$V = \{(x_1, x_2, x_3, x_4)^t \,:\, x_1 - 2x_3 + x_4 = 0\}$$

Show that V is a subspace. Exhibit a basis of V.

(12) Let W be a subspace of a vector space V (over a field F). Given $v \in V$ define

$$v + W = \{v + w \,:\, w \in W\}.$$

The subset $v + W$ is called a coset of the the subspace W. Let $u, v \in V$.

(a) Show that $u + W = v + W$ if and only if $u - v \in W$.

(b) Let V/W be the set of all cosets of W in V. In other words

$$V/W = \{v + W \,:\, v \in V\}.$$

Given two cosets $u + W, v + W \in (V/W)$ define their addition to be

$$(u + W) + (v + W) = (u + v) + W$$

and given a scalar $a \in F$ define

$$a \cdot (v + W) = (av) + W.$$

Show that with this definition of addition and scalar multiplication, V/W become a vector space over F called the quotient space of V mod W.

(13) Show that the intersection of any two planes through the origin in $\mathbb{R}^3$ contains a line through the origin.

(14) Suppose that U, U', U'' are subspaces of V such that

$$U \oplus U' = U \oplus U''.$$

Does it follow that $U' = U''$?

(15) Let $T : V \longrightarrow W$ be an isomorphism of vector spaves. Show that $T(-v) = -T(v)$.

(16) Let V, W be finite dimensional vector spaces over F. Suppose that V is not isomorphic to any subspace of W. Show that $\dim(V) > \dim(W)$.

(17) Let $0 \neq a \in \mathbb{R}$. Show that the map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $T(x_1, \dots, x_n) = (ax_1, x_2, \dots, x_n)$ is an isomorphism.

(18) Show that every vector space over $\mathbb{C}$ is a vector space over $\mathbb{R}$. If V has basis $(v_1, \dots, v_n)$ over $\mathbb{C}$ exhibit a basis over $\mathbb{R}$.

(19) Let $T, S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear operators

$$T(x_1, x_2)^t = (2x_1 + 3x_2, x_1 - x_2)^t, \quad S(x_1, x_2)^t = (x_1, 2x_1 - 5x_2)^t.$$

Find the matrices of $T, T + S, S \circ T, T \circ S, 3T$ relative to the standard basis.

(20) Let W be a subspace of V. Show that the map $T : V \longrightarrow V/W$ defined by $T(v) = v + W$ is a linear map. Describe its image and kernel. If V is finite dimensional show that so is V/W.

(21) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a surjective linear map. Show that there is a linear map $S : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ such that $T \circ S$ is the identity linear transformation. The corresponding statement for matrices is the following. If A is a $m \times n$ matrix and $\operatorname{rank}(A) = m$, then there exists a $n \times m$ matrix C such that CA is the identity $n]timesn$ matrix.

(22) Suppose A is a $m \times n$ matrix with $\operatorname{rank}(A) = n$. Show that there exists a $n \times m$ matrix D such that AD is the identity matrix. Make a corresponding statement for linear maps.

(23) Show that every linear transformation T can be written as a composition of two linear maps one of which is injective and the other surjective.

(24) Find all matrices that commute with

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(25) A $n \times n$ matrix A is said to be symmetric if $A = A^t$. Show that the set Sym_n of all $n \times n$ symmetric matrices is subspace of $M_n(\mathbb{R})$. Find the dimension of Sym_n. Find a subspace V such that

$$\operatorname{Sym}_n \oplus V = M_n(\mathbb{R}).$$

(26) Let $\operatorname{tr}(A)$ denote the trace of the (square) matrix A. Show that if $\operatorname{tr}(AB) = 0$ for all matrices B, then A is the zero matrix.

(27) Let $T : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ be a linear map with the property that $T(A) = 0$ whenever $A^2 = 0$. show that there exists a scalar a such that $T(A) = a \cdot \operatorname{tr}(A)$ for all A.

(28) Let $E = (e_1, e_2, e_3)$ be the standard basis of $\mathbb{R}^3$. Find the coordinate vector of $(1, 2, 3)^t$ in the basis $B = (e_1 + e_2, e_2, e_3)$.

(29) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear map with matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

in the standard basis. Find the matrix of T in the basis $B = (e_1 + e_2, e_2, e_3)$.

(30) Let A be a $m \times n$ matrix with $\operatorname{rank}(A) = m$. Show that there exists a $n \times m$ matrix B such that AB is the identity $m \times m$ matrix. Formulate this in terms of linear transformations.

(31) Let A be a $m \times n$ matrix with $\operatorname{rank}(A) = n$. Show that there is a $n \times m$ matrix C such that CA is the identity $n \times n$ matrix.

(32) Show that every linear transformation can be written as the composition of two linear transformations one of which is injective and the other surjective.

(33) Let A be a matrix of rank r. How does the rank change when exactly one entry of A is altered. When exactly two entris are altered.

(34) Show that $\operatorname{rank}(A) \leq 1$ if and only if there exist column vectors x, y such that

$$A = xy^t$$

. Show that equality holds if and only if both x and y are nonzero.

(35) Does every non zero column vector have a left inverse?

(36) Let A be an invertible $n \times n$ matrix. Let B be any $n \times r$ matrix. Show that the $n \times (n + p$ matrix

$$(A\ B)$$

has a right inverse.

(37) Exhibit two right inverses of the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$$

(38) If B, C are two left inverses of A show that calar.

$$aB + (1 - a)C$$

is also a left inverse of A where a is a scalar.

(39) Prove or disprove : If A is a $m \times n$ matrix of rank m and B is a $n \times m$ of rank m, then AB is invertible.

(40) Show that for any square matrix A (with real entris) there is a scalar a such that

$$aI + A$$

is invertible. Show that this is false over the field F_2.

(41) Describe all 2×2 matrices with real entries such that $A^{-1} = A$.

(42) Let A be an idempotent matrix (that is, $A^2 = A$). If A is invertible, show that $A = I$.

(43) Show that if $A^2 = A^3$ and $\operatorname{rank}(A = \operatorname{rank}(A^2)$, then $A^2 = A$.

(44) Let $T, S : V \longrightarrow W$ be two linear transformations. Show that $\operatorname{im}(S) \subseteq \operatorname{im}(T)$ if and only if $\ker(T) \subseteq \ker(S)$.

(45) If $A^2 = A$, then show that $\operatorname{rank}(A) = \operatorname{tr}(A)$.

(46) If A, B are matrices of the same size, show that

$$\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

(47) Let A be a real $m \times n$ matrix and let v be a linear combination of the columns of A. Show that u is a solution of the system $AX = v$ if and only if u is a solution of the system

$$A^t AX = A^t v.$$

(48) For any real matrix $m \times n$ matrix A and for any $m \times 1$ column vector B, show that the system show that the system

$$A^t AX = A^t B$$

is consistent.