LINEAR ALGEBRA I - TEST I

Answer all questions. You may use results proved in class after correctly quoting them. Any other claim must be accompanied by a complete proof. Upload your answers to Moodle by 5:45 PM. Submissions will close at 5:45 PM. Maximum marks is 20 and all questions carry equal marks.

(1) Solve the system of equations

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 1 \\ 3x_1 + 4x_4 &= 1 \\ x_1 - 4x_2 - 2x_3 + 2x_4 &= 0 \end{aligned}$$

by transforming the augmented matrix into its row echelon form.

(2) Suppose that a system of linear equations $AX = B$ over the reals has a unique solution for some particular column vector B. Show that the system has a unique solution for all column vectors B.

(3) Let V be the set of all positive real numbers. Given $x, y \in V$ and $a \in \mathbb{R}$ define addition (denoted by $+$) and scalar multiplication (denoted by $\odot$) in V by the rules

$$x + y = x \cdot y, \quad a \odot x = x^a$$

where $x \cdot y$ is the usual multiplication in $\mathbb{R}$. Show that with this definition of addition and scalar multiplication V becomes a vector space over $\mathbb{R}$. Decide whether V and $\mathbb{R}$ are isomorphic as vector spaces over $\mathbb{R}$. [3+2]

(4) Let $P_3(x)$ denote the vector space over $\mathbb{R}$ of polynomials with real coefficients of degree at most 3. Let $a < b$ be two real numbers and let

$$V = \{p(x) \in P_3(x) \,:\, p(a) = 0 = p(b)\}.$$

Then V is a subspace of $P_3(x)$. Find the dimension of V over $\mathbb{R}$.