LINEAR ALGEBRA I - TEST II

Answer all questions. You may use results proved in class after correctly quoting them. Any other claim must be accompanied by a complete proof. Upload your answers to Moodle by 6:15 PM. Submissions will close at 6:15 PM. Maximum marks is 20.

(1) Prove that a vector space V is infinite dimensional if and only if there exists a sequence of vectors $v_1, v_2, \ldots$ in V such that $(v_1, v_2, \ldots, v_n)$ is linearly independent for each integer n. [4]

(2) Let $T : V \longrightarrow V$ be a linear map. A subspace $W \leq V$ is said to be invariant if $T(W) \subseteq W$. Here $T(W)$ denotes the image of T. Give an example of an operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that T has no nontrivial invariant subspace. Recall that a subspace W of V is nontrivial provided $W \neq \{0\}, V$. [4]

(3) Let V, W be finite dimensional vector spaces. Let $U \leq V$ be a subspace. Prove or (give an example to) disprove each of the following statements. Justify.
(a) There exists a surjective linear map $T : \mathrm{Hom}(U, W) \longrightarrow \mathrm{Hom}(V, W)$.
(b) There exists a surjective linear map $S : \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(U, W)$. [4+4]
Recall that $\mathrm{Hom}(V, W)$ denotes the vector space of all linear transformations from V to W.

(4) Let P denote the vector space of real polynomials of degree at most 3. Find the matrix of the linear map

$$\frac{d}{dx} : P \longrightarrow P; \quad \frac{d}{dx}(p(x)) = p'(x)$$

relative to the basis $B = (1, 1 + x, x^2, x^3)$. Here $p'(x)$ is the derivative of $p(x)$. [4]