Probability I: Additional Problems

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The problems are taken from various sources (Feller, Santosh S. Venkatesh, Pitman, Ross, K. L. Chung et al.) and often copied verbatim. These are for additional practice by students. I apologise for not citing the sources always.

Please always write the sample spaces and practice. These will be constantly updated and could be from future material too. Please do not post solutions on the class forum until a week after the corresponding material is covered.

1. In Ex. 5(b) of Assignment 1, let F be the event that there are three mutual friends and F' be the event that there are three people such that no two of them are friends. Show that $\Omega \neq F \cup F'$. However, if we take a group of six people then show that $\Omega = F \cup F'$.

2. Let P be polynomials of degree d and $n \in \mathbb{N}$. Let the outcome of an experiment be the zeros of P in $[n]$. Describe the sample space Ω and find its cardinality.

3. Six cups and six saucers come in pairs, two pairs are red, two are white, and two are blue. If cups are randomly assigned to saucers find the probability that no cup is upon a sacuer of the same colour.

4. A lottery specifies a random subset R of r out of the first n natural numbers by picking one at a time. Determine the probabilities of the following events: (a) there are no consecutive numbers, (b) there is exactly one pair of consecutive numbers, (c) the numbers are drawn in increasing order. Suppose that you have picked your own random set of r numbers. What is the probability that (d) your selection matches R?, (e) exactly k of your numbers match up with numbers in R ?

5. Each of n sticks is broken into a long part and a short part, the parts jumbled up and recombined pairwise to form n new sticks. Find the probability (a) that the parts will be joined in the original order, and (b) that all long parts are paired with short parts. [1]

[1] If sticks represent chromosomes broken by, say, X-ray radiation, then a recombination of two short or two long parts can cause cell death. See Problem II.10.12 of Feller's book.

6. [Spread of rumours:] In a small town of n people a person passes a titbit of information to another person. A rumour is now launched with each recipient of the information passing it on to a randomly chosen individual. What is the probability that the rumour is told r times without (a) returning to the originator, (b) being repeated to anyone. Generalisation: redo the calculations if each person tells the rumour to m randomly selected people.

7. [The hot hand:] A particular basketball player historically makes one basket for every two shots she takes. During a game in which she takes very many shots there is a period during which she seems to hit every shot; at some point, say, she makes five shots in a row. This is clearly evidence that she is on a purple patch (has a "hot hand") where, temporarily at least, her chances of making a shot are much higher than usual, and so the team tries to funnel the ball to her to milk her run of successes as much as possible. Is this good thinking? Propose a model probability space for this problem and specify the event of interest.

8. Let k be a natural number and $\Omega_n = [n]$ be the sample space and the probability be the uniform distribution on $[n]$ i.e., equally likely outcomes. Denote by A_n the event consisting of multiples of k. What is $\mathbb{P}(A_n)$ for $n = mk$ for some $m \geq 1$? What is the limit of probability $n^{-1}\mathbb{P}(A_n)$?

9. An urn contains n red and m blue balls. They are withdrawn one at a time randomly and without replacement until a total of $r, r \leq n$ red balls have been withdrawn. Find the probability that a total of k balls are withdrawn.

10. In a season, a team has n wins and m losses. A run of wins is a successive sequence of wins. For example, if the team has WWLLLWLLWW then it has 3 runs of wins and 2 runs of losses. Assuming that all possible orderings of wins and losses are equally likely, compute the probability that team has a total of k runs (i.e., number of runs of wins + number of runs of losses).

11. Find the correct solution to the problem of points. Also explain why does Fermat's solution not give an error.

12. Let $\mathbb{P}_p$ denote the $Bin(n, p)$ probability distribution where $p \in [0, 1]$ and the sample space is $\Omega = [n] \cup \{0\}$. Define $A_k = \{k, k+1, \ldots, n\}$. Show that for all $k \in \Omega$, $\mathbb{P}_p(A_k) \leq \mathbb{P}_{p'}(A_k)$ if $p \leq p'$.

13. In an exam there are five multiple-choice questions and each with four possible choices. A clueless student writes the exam and chooses answers at random. Compute the probability that he gets at least 3 out of five questions correct in the following two scenarios.

    (a) The student is informed that the first three questions have only one correct answer and the last two questions have two correct answers each and in the latter questions both the correct answers need to be chosen.

(b) The student is informed that there are three questions (without specifying the question numbers) with only one correct answer and the remaining three questions have two correct answers each and in the latter questions both the correct answers need to be chosen

14. Three players a, b, and c take turns at a game in a sequence of rounds. In the first round a plays b while c waits for her turn. The winner of the first round plays c in the second round while the loser skips the round and waits on the sideline. This process is continued with the winner of each round going on to play the next round against the person on the sideline with the loser of the round skipping the next turn. The game terminates when a player wins two rounds in succession. Suppose that in each round each of the two participants, whoever they may be, has probability 1/2 of winning unaffected by the results of previous rounds. Let x_n, y_n and z_n be the conditional probabilities that the winner, loser, and bystander, respectively, in the nth round wins the game given that the game does not terminate at the nth round. (a) Show that $x_n = \frac{1}{2} + \frac{1}{2}y_{n+1}, y_n = \frac{1}{2}z_{n+1}$ and $z_n = \frac{1}{2}x_{n+1}$. (b) By a direct argument conclude that, in reality, $x_n = x, y_n = y$ and $z_n = z$ are independent of n and determine them. (c) Conclude that a wins the game with probability 5/14.

15. Imagine there are n people in line to board a plane that seats n people. The first person in line realizes he lost his boarding pass so when he boards he decides to take a random seat instead. Every person that boards the plane after him will either take their "proper" seat, or if that seat is taken, a random seat instead. What is the probability that the last person that boards will end up in his/her proper seat ?

16. Consider an island with $n + 2$ inhabitants. One of them is killed, and the murderer must be one of the inhabitants of the island. Police investigators discover a DNA profile at the scene of the crime. Scientists are able to say that this particular DNA profile occurs in a fraction p of all people. Now the police starts a big screening of all the inhabitants of the island. The first person to be screened, let's call him John Smith, turns out to have this particular DNA profile. What is the probability that John Smith is the murderer? Think of a mathematical formulation before reading below.

    Here is a more mathematical formulation of the problem :

    Let us say there are k gene types and each person has gene type i with probability p_i and independent of other persons. That is the gene types of the $n+1$ people can be represented as a vector $(u_1, \ldots, u_{n+1})$ with the probability being a product probability and for each individual coordinate j the probabilities are $\mathbb{P}(u_j = i) = p_i$where $p_i \geq 0 \forall i$and $p_1 + \ldots + p_k = 1$.

    Assume that person 1 is the murderer and we know his gene type, say 1 without loss of generality. Let the first person be chosen uniformly at random and independent of the gene types. Call the index of the person as J.

    The events under question are

$$G = \{\text{Person } J \text{ is the murderer}\} = \{J = 1\}$$

$E = \{\text{Person } J\text{'s gene type is same as that of the murderer}\} = \{u_J = u_1 = 1\}$.

The question is to compute $P(G|E) = P(J = 1|u_J = u_1 = 1)$.

A second question can be to compute $P(J = 1|u_J = u_1)$.

17. A prisoner is given an interesting chance for parole. He's blindfolded and told to choose one of two bags; once he does, he is to reach in and pull out a marble. If he pulls out a red marble he is set free; if it's a black, his parole is denied. In total, there are 50 red marbles and 50 black marbles. The prisoner can choose to distribute the marbles in the bags. Suppose he choses to put r red marbles and b black marbles in the first bag. What is the r, b that maximizes his chance of being freed ?