2/11 LECTURE 10 - CONDITIONAL PROBABILITY.

1980/81 Wimbledon final. — Prob. (Bjorg winning) $\approx 2/3$.
"Survey"

Prob (Bjorg loses 1^{st} set & loses the match) $\approx 1/2$.

$>$ Prob (loses the match) $= 1/3$.

Wrong! Most people thought
" Prob. Bjorg loses the match given he loses the first set"?

Prob. of Kohli winning the toss against Afghanistan given he lost both the previous tosses = ??.

$\Omega = \{0, 1\}^3$

$A = \{(0,0,c) : c \in \{0,1\}\}$ — lost first two tosses.

$B = \{(a,b,1) : a,b \in \{0,1\}\}$ — win 3rd toss.

0: lose the toss; 1: win the toss

Assume eq. likely outcomes. — $P(B) = 1/2$; $P(A) = 1/4$.

$\mathbb{P}(B \text{ given } A) =: \mathbb{P}(B|A)$

Since A is given, not all outcomes in Ω are valid anymore.

So "new sample space is A"! May be $\mathbb{P}(B|A)$ "$\cancel{=}$" $\mathbb{P}(B \cap A)$

2 problems — $\mathbb{P}(B|A)$ $\forall B \subseteq A$ is not a prob. dist. on A.

$(\mathbb{P}(A|A) = \mathbb{P}(A) \underset{(?,)}{<} 1)$

— $\mathbb{P}(B|A) < \mathbb{P}(A), \mathbb{P}(B)$. — doesn't go with heuristic understanding.

DEFN 10.1 | $(\Omega, \mathbb{P})$ be a P-space. Let $B \subseteq \Omega$ be $\ni \mathbb{P}(B) > 0$.

Then $\mathbb{P}(A|B) := \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ — Conditional prob. of A given B

or prob of A given B.

or prob. of A conditioned on B.

Ex. 10.2: Consider eg. of Kohli winning the toss. ["Gambler's fallacy"]

$$\mathbb{P}(\underbrace{B}_{\text{win } 3^{rd} \text{ toss}} \mid \underbrace{A}_{\text{lose first two tosses}}) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{|A \cap B| / |\Omega|}{|A| / |\Omega|} = \frac{1}{2}.$$

(Prod. of Bernoulli PD)

Ex*: 10.3

$\Omega = \{0,1\}^5$ $\quad \mathbb{P}_p(\omega) = \prod_{i=1}^{5} p^{\omega_i} (1-p)^{1-\omega_i}$ $\quad \omega = (\omega_1, \dots, \omega_5) \in \Omega.$

$A_i = \{\omega : \omega_i = 1\}$ $\quad$ prob. of Bjorg winning a set. $\leftarrow p \in [0,1]$

check: $\mathbb{P}_p(A_i) = p$.

B = winning the match. (wins at least 3 sets) $\quad \mathbb{P}_p(B^c \mid A_1^c) > \mathbb{P}_p(B^c)$ possible for some p?

$$[\mathbb{P}_p(B^c \cap A_1^c) \leq \mathbb{P}_p(B^c) \quad \forall p]$$

Ex 10.4: Let $B \subseteq \Omega$ & $\mathbb{P}(B) > 0$. Define $\mathbb{P}_B(A) := \mathbb{P}(A \mid B) \quad \forall A \subseteq B$.

(A) Show that $(B, \mathbb{P}_B)$ is a P. space i.e.,

(i) $\mathbb{P}_B(A) \geq 0 \quad \forall A \subseteq B$ $\quad$ (ii) $\mathbb{P}_B(B) = 1$ $\quad$ (iii) $\mathbb{P}_B(A \sqcup C) = \mathbb{P}_B(A) + \mathbb{P}_B(C)$. (i.e., $A \cap C = \phi$).

Ex. 10.5 $(\Omega, \mathbb{P}_B)$ is a P. space where $\mathbb{P}_B(A) = \mathbb{P}(A|B)$ $\forall A \subseteq \Omega$.

Eg 10.6 Let $(\Omega, \mathbb{P})$ be P.S. with eq. likely outcomes in Ω.

i.e., $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$

Assume $|B| > 0$ i.e., $\mathbb{P}(B) > 0$.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{|A \cap B|}{|B|}$$

$\Rightarrow$ $\mathbb{P}_B(A) = \frac{|A|}{|B|}$ if $A \subseteq B$.

$\Rightarrow$ $(B, \mathbb{P}_B)$ is a P.S. with eq. likely outcomes in B.

Eg 10.7.

A house with 2 children is chosen at random & you are told that there is a boy. what is the prob. that the other child is a boy?

Soln 1: $\Omega = \{bb, bg, gb, gg\}$; $P(\omega) = \frac{1}{4}$ $\forall\, \omega \in \Omega$.

$B = \{\exists \text{ a boy}\} = \{bb, bg, gb\}$ $\quad P(B) = 3/4$

$A = \{\text{both are boys}\} = \{bb\}$

$P(A|B) = \frac{|A|}{|B|} = \frac{1}{3}$.

Soln. 2: $\Omega = \{bb, \underset{\underset{gb}{\parallel}}{bg}, gg\}$ $\quad p(\omega) = \frac{1}{3}$ $\forall\, \omega \in \Omega$.

$B = \{bb, bg\}$ $\quad A = \{bb\}$ $\quad P(A|B) = \frac{1}{2}$.

Soln 3: $\Omega = \{bb, bg, gg\}$ $\quad p(bb) = p(gg) = \frac{1}{4}$; $p(bg) = 1/2$.

$\mathbb{P}(B) = 3/4$; $\mathbb{P}(A \cap B) = 1/4$.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = 1/3.$$ (Same as first soln)

Ex* $p(gg) = p$, $p(bb) = q$, $p+q \le 1$. Solve the problem.

10.8

Eg 10.9 Same as above. You are told the eldest child is a boy. What is Prob. both are boys?

$\Omega = \{bb, bg, gb, gg\}$ $p(\omega) = \frac{1}{4}$ $\forall\, \omega \in \Omega$.

$B = \{bb, bg\}$ eldest child is a boy.

$A = \{bb\}$ both are boys. $\mathbb{P}(A|B) = \frac{1}{2}$.

Eg 10.10 Same as above. You call a house with 2 children, a boy picks the phone & puts it down. Prob. both are boys?

$$\Omega = \{bb, gb, bg, gg\} \times \{b, g\}$$

children in house $\qquad$ who picks up the phone.

$$\mathbb{P}((gg, b)) = 0 = \mathbb{P}((bb, g))$$

$$\mathbb{P}((gg, g)) = \frac{1}{4} = \mathbb{P}((bb, b))$$

$$\mathbb{P}((gb, *)) = \frac{1}{8} = \mathbb{P}((bg, *)) \qquad * \in \{b, g\}$$

B = two children & boy picks up the phone

$$= \{(bb, b), (gb, b), (bg, b)\} \qquad \mathbb{P}(B) = \frac{1}{2}.$$

A = both are boys = $\{(bb, b)\}$

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) = \frac{1}{4}. \qquad \mathbb{P}(A \mid B) = 1/2. \qquad \square$$

THM 10.11: Let $A_1, \ldots, A_n$ be events in Ω $\ni$ $\mathbb{P}(A_1 \cap \cdots \cap A_n) > 0$.

Then $\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1 | A_2 \cap \cdots \cap A_n) \times \mathbb{P}(A_2 | A_3 \cap \cdots \cap A_n)$

$\times \cdots \times \mathbb{P}(A_{n-1} | A_n) \times \mathbb{P}(A_n)$.

(chain rule for condnl. prob.)

Proof: By induction. For $n=1$ $\mathbb{P}(A_1) = \mathbb{P}(A_1)$

For $n=2$ $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1 | A_2) \times \mathbb{P}(A_2)$

(defn of $\mathbb{P}(A_1 | A_2)$)

Suppose holds for $n-1$.

Since $\mathbb{P}(A_1 \cap \cdots \cap A_n) > 0$, all condnl. prob. are well-defined.

$B = A_2 \cap \cdots \cap A_n$

$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1 \cap B) \overset{\text{by defn}}{=} \mathbb{P}(A_1 | B)\ \mathbb{P}(B)$

(by indn) $= \mathbb{P}(A_1 | B)\ \mathbb{P}(A_2 | A_3 \cap \cdots A_n) \times \cdots \times \mathbb{P}(A_{n-1} | A_n) \times \mathbb{P}(A_n)$

$\blacksquare$

THM 10.12. Let $\Omega = \bigsqcup_{i=1}^{n} A_i$ & $\mathbb{P}(A_i) > 0 \quad \forall i.$

Then $\forall \; B \subseteq \Omega \quad \mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \mid A_i)\,\mathbb{P}(A_i).$

[Law of Total prob.]

Try A4.9.