11/11 LECTURE 12 : EXAMPLES FOR CONDNL. PROB. & BAYES FORMULA.

Eg 12.1 : A plane goes missing in one of 3 possible regions. Assume u.A.R.
If a plane is missing in region i, a search of region i is successful w.p. $1-\beta_i$. (β_i - overlook prob.)
What is prob. plane is in region i given search in region 1 is unsuccessful?

Soln:

$$\Omega = \underbrace{\{1,2,3\}}_{\text{region where plane is there.}} \times \{F, S\}$$

F → search in region 1 is unsuccessful.
S → search in region 1 is successful.

$R_i = \{i\} \times \{F, S\}$ — plane is in region i.

$F = [3] \times \{F\}$ - search in region 1 is unsuccessful.

$\omega \in \Omega \Rightarrow \omega = (i, *)$

i → missing region
$*$ → F/S after search in region 1.

$\mathbb{P}(R_i \mid \bar{F})$ = Prob. plane is in region i if search in region 1 is unsuccessful.

$$\mathbb{P}(R_1 \mid \bar{F}) = \frac{\mathbb{P}(\bar{F} \mid R_1)\,\mathbb{P}(R_1)}{\sum_{j=1}^{3} \mathbb{P}(\bar{F} \mid R_j)\,\mathbb{P}(R_j)}$$

$$= \frac{\beta_1}{\beta_1 + 2} \leftarrow \begin{cases} \mathbb{P}(\bar{F} \mid R_1) = \beta_1 \\ \mathbb{P}(\bar{F} \mid R_i) = 1 \quad i = 2,3. \\ \mathbb{P}(R_i) = 1/3 \quad i = 1,2,3. \end{cases}$$

$i = 2,3$ $$\mathbb{P}(R_i \mid \bar{F}) = \frac{1}{\beta_1 + 2}.$$

$$\mathbb{P}(\bar{F}) = \frac{\beta_1 + 2}{3} = \sum_{j=1}^{3} \mathbb{P}(\bar{F} \mid R_j)\,\mathbb{P}(R_j).$$

Eg 12.2: Choose a permutation from S_n U.A.R.

What is the prob. of exactly k matches? $\sigma(i) = i$ exactly for k indices

Soln:

$P_n(k)$ = Prob. of exactly k matches. (Using I-E)

$$P_n(0) = \sum_{i=0}^{n} \frac{(-1)^i}{i!} \quad \text{(A4 / class)}$$

Fix k indices $i_1, \dots, i_k$. $E(i_1, \dots, i_k) = \{\sigma(i_j) = i_j$ in the given k indices$\}$

$F(i_1, \dots, i_k) = \{$No matches in the remaining $n-k$ indices$\}$

$$P_n(k) = \sum_{i_1, \dots, i_k}^{*} \mathbb{P}(E(i_1, \dots, i_k) \cap F(i_1, \dots, i_k)) \quad \text{(fin. additivity)}$$

Fix $i_1, \dots, i_k$ & set $E = E(i_1, \dots, i_k)$

& $F = F(i_1, \dots, i_k)$

$$\mathbb{P}(E \cap F) = \mathbb{P}(F \mid E)\,\mathbb{P}(E).$$

$$(*) \quad \mathbb{P}(E) = \frac{1}{N} \cdot \frac{1}{(N-1)} \cdots \frac{1}{(N-k+1)} = \frac{(N-k)!}{N!}$$

(Ex.) $\mathbb{P}(F \mid E)$ = Prob. there are no matches in remaining $N-k$ indices

$= P_{N-k}(0)$

$$P_N(k) = \frac{1}{k!} P_{N-k}(0) = \frac{1}{k!} \sum_{i=0}^{N-k} \frac{(-1)^i}{i!}$$

Eg 12.3 Monty hall problem. – There are 3 closed doors. Prize behind one of the doors. Zonks (0 value items) behind other two. Randomly arranged. Contestant picks one door at random. Host (M. H.) opens one of the other two doors & shows a zonk. MH asks if you want to change your door? Should you change or not?

Soln:

prize — P, zonks — α, α

$\Omega = \{P\alpha\alpha, \alpha P\alpha, \alpha\alpha P\} \times \{1, 2, 3\}$

arrangement of prize.

↳ door contestant chooses

$P(\omega) = \frac{1}{9}$. $\forall \, \omega \in \Omega$.

P_i — Prize is in door i ; D_i — Cont. picks door i

$\mathbb{P}(P_1 | D_1) = \frac{\mathbb{P}(D_1 \cap P_1)}{\mathbb{P}(D_1)} = \frac{1/9}{1/3} = \frac{1}{3}$.

prize is in door 1. contestant picks door 1

$\mathbb{P}(P_2 \cup P_3 | D_1) = \frac{2}{3}$ (Prob. prize is in one of the other doors.)

Answer is same $\forall \, i = 1, 2, 3$

Prob. of winning if you change $= 2/3$.

Eg 12.4: Shuffle a deck of cards i.e., pick a permutation in S_{52} UAR. You are shown cards one by one. You've to guess A spade. Before first card, you have to guess if it is a A sp. If you guess correctly, you win. Repeat it.

Soln: Suppose you guess up-front - i.e., choose some card i with $p(i)$ - prob of i

$$\sum_{i=1}^{52} p(i) = 1. \quad p(\cdot) \text{ is a pmf on } [52].$$

$$\mathbb{P}(\text{winning}) \underset{\text{LTP}}{=} \sum_{i=1}^{52} \underbrace{\mathbb{P}(A\ sp \text{ is in } i^{th} \text{ position} \mid \text{you've chosen } i)}_{\mathbb{P}(\ \downarrow\ )}$$

$$= \sum_{i=1}^{52} \frac{1}{52} p(i) = \frac{1}{52}.$$

Is there a better strategy?