16/11 LECTURE 13 — INDEPENDENCE.

Let A & B be 2 events. chance of Outcome in A isn't affected by outcome being in B & vice-versa. We say A & B are "independent". How to express mathematically?

Eg 13.1 Pick a card at random from a standard deck.

A = card is a spade $\quad \mathbb{P}(A) = \frac{1}{4}$

B = card is A, 2, ..., 10. $\quad \mathbb{P}(B) = \frac{10}{13}$

$$\mathbb{P}(A|B) = \frac{1}{4} = \mathbb{P}(A)$$

$$\mathbb{P}(B|A) = \frac{10}{13} = \mathbb{P}(B)$$

$$\Rightarrow \underbrace{\mathbb{P}(A \cap B)}_{= \frac{10}{52}} = \mathbb{P}(A)\,\mathbb{P}(B).$$

Eg 13.2 Shuffle a standard deck of cards.

A = first card is a spade. $\quad \mathbb{P}(A) = 1/4$

B = second card is a spade. $\quad \mathbb{P}(B) = 1/4$

$$\mathbb{P}(B|A) = \frac{12}{51} < \frac{1}{4} = \mathbb{P}(B).$$

Knowing A makes B less likely!

Reasonable to say A & B are "indep." if $\mathbb{P}(A|B) = \mathbb{P}(A)$ & $\mathbb{P}(B|A) = \mathbb{P}(B)$.

If $\mathbb{P}(A), \mathbb{P}(B) > 0$ then $\mathbb{P}(A|B) = \mathbb{P}(A) \iff \mathbb{P}(B|A) = \mathbb{P}(B)$

$\iff \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

Defn 13.3: $A, B \subseteq \Omega$ are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Eg 13.4: Let $A \cap B = \emptyset$ Then A & B are indep. iff $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

Eg 13.5: If $\mathbb{P}(A) = 0$ & $B \subseteq \Omega$ then A & B are independent.

Eg 13.6: If $A = \Omega$ then A & B are indep. $\forall\ B \subseteq \Omega$.

Eg 13.7: If A & B are indep. then so are A^c & B, A & B^c, A^c & B^c.

Proof: $\mathbb{P}(A^c \cap B) = \mathbb{P}(B / A\cap B) = \mathbb{P}(B) - \mathbb{P}(A\cap B)$ $(A\cap B \subseteq B)$

$\underset{(\text{indep})}{=} \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(B)(1-\mathbb{P}(A))$

$= \mathbb{P}(B)\,\mathbb{P}(A^c)$.

others can be proved similarly.

Defn 13.8 $A_1, \dots, A_n$ are pairwise indep. if $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$ $\forall i \neq j$.

$A_1, \dots, A_n$ are (mutually) indep. if $\mathbb{P}\left(\bigcap_{i\in J} A_i\right) = \prod_{i\in J} \mathbb{P}(A_i)$ — (Ind)

$\forall\, \phi \neq J \subseteq [n]$.

(Trivially) Indep $\Rightarrow$ pairwise indep.

Ex. 13.9. Construct an example of 3 events that are pairwise indep but not indep.

Lemma 13.10 If $A_1, \dots, A_n$ are independent then so are $A_1^*, \dots, A_n^*$ where $A_i^* = A_i$ or A_i^c.

Proof: One case. $A_1^* = A_1, \dots, A_{n-1}^* = A_{n-1}$ & $A_n^* = A_n^c$.

$$\mathbb{P}(A_1 \cap \dots \cap A_n^c) = \mathbb{P}(A_1 \cap \dots \cap A_{n-1} / A_1 \cap \dots \cap A_n)$$

$$= \mathbb{P}(A_1 \cap \dots \cap A_{n-1}) - \mathbb{P}(A_1 \cap \dots \cap A_n)$$

(indep) $$= \prod_{i=1}^{n-1} \mathbb{P}(A_i) - \prod_{i=1}^{n} \mathbb{P}(A_i)$$

(Ind) $$= \prod_{i=1}^{n-1} \mathbb{P}(A_i)\, \mathbb{P}(A_n^c).$$

shows, for $J = [n]$. Verify $\forall\, J \subseteq [n]$, $J \neq \phi$ & also other cases. ∎

LEMMA 13.1.1 Let $(\Omega, \mathbb{P}) = (\Omega_1 \times \dots \times \Omega_n, \mathbb{P}_1 \times \dots \times \mathbb{P}_n)$ i.e., (Prod. prob. space)

$$p(\omega) = p_1(\omega_1) \cdots p_n(\omega_n) \quad \text{for } \omega = (\omega_1, \dots, \omega_n)$$

Let $A_i \subseteq \Omega_i$, $i = 1, \dots, n$. $\overline{A_i} = \Omega_1 \times \dots \times A_i \times \dots \times \Omega_n \subseteq \Omega$.

Then $\overline{A_1}, \overline{A_2}, \dots, \overline{A_n}$ are indep. [Abuse of notation - we say $A_1, \dots, A_n$ are indep.]

Proof: Let me check (Ind) condition for $J = [n]$ & rest is an exercise.

$$\bigcap_{j=1}^{n} \bar{A_j} = A_1 \times \cdots \times A_n .$$

Prop. of prod. prob is $\mathbb{P}(\bar{A_i}) = \mathbb{P}_i(A_i) \quad \forall i$ [A 2]

$$\mathbb{P}(A_1 \times \cdots \times A_n) = \prod_{i=1}^{n} \mathbb{P}_i(A_i)$$

So $\mathbb{P}(\bigcap_{j=1}^{n} \bar{A_j}) = \mathbb{P}(A_1 \times \cdots \times A_n) = \prod_{j=1}^{n} \mathbb{P}_j(A_j) = \prod_{j=1}^{n} \mathbb{P}(\bar{A_j})$. $\boxtimes$

Terminological: $(\Omega_i, \mathbb{P}_i)$ — Experiment 'i'. $i = 1, \ldots, n$.

If we say, Experiment $1, \ldots, n$ are performed independently then we refer to the prod. prob. space $(\Omega, \mathbb{P}) = (\Omega_1 \times \cdots \times \Omega_n, \mathbb{P}_1 \times \cdots \times \mathbb{P}_n)$.

Eg 13.12 $\Omega_i = \{0,1\}$ $\mathbb{P}_i(1) = p_i$ $1 \le i \le n$.

Experiment i is coin toss with heads prob 'p_i'

Let us assume the n coins are tossed independently.

Then the prob. space is $\Omega = \Omega_1 \times \cdots \times \Omega_n = \{0,1\}^n$

$$\mathbb{P}(\omega) = \prod_{i=1}^{n} p_i^{\omega_i}(1-p_i)^{1-\omega_i} \quad \omega = (\omega_1, \ldots, \omega_n) \in \{0,1\}^n.$$

Eg 13.13 (1) n coins (all with heads prob. p) are tossed independently.

Prob. space is ?

(2) Toss a fair coin, roll a dice & pick a card independently.

(3) Shuffle two decks of cards independently.

(4) Pick 'r' cells from 'n' cells independently i.e.; pick a cell from 'n' cells w.a.r & repeat this exp. 'r' times indep.!