18/11 LECTURE 14 — INDEPENDENCE — EXAMPLES.

Eg 14.1. Pick 'r' cells from 'n' cells independently i.e; pick a cell from 'n' cells u.a.r & repeat this exp. 'r' times indep.!

$\Omega = [n]^r$ $\quad p(\omega) = \frac{1}{n^r}$ $\quad \omega = (\omega_1, \ldots, \omega_r) \in \Omega.$

Eg 14.2: Special cases of Eg 14.1

(1) $n=2$. $\rightarrow$ r indep. fair coin tosses (r indep. bits)

(2) $n=6$ $\rightarrow$ r indep. die rolls.

(3) $n \geq 1$ $\rightarrow$ r persons indep. pick a hash-code from $1, \ldots, n$.

Eg 14.3: n persons pick indepently a no: whereby i^{th} person picks a no: in $[i]$ U.A.R.

$$\Omega = \prod_{i=1}^{n} [i] \quad ; \quad P(\omega) = \prod_{i=1}^{n} \frac{1}{i} = \frac{1}{n!} \qquad \omega = (\omega_1, \dots, \omega_n) \in \Omega.$$

$$= \prod_{i=1}^{n} \{1, \dots, i\} = \{1\} \times \{1,2\} \times \{1,2,3\} \times \cdots \times \{1, 2, \dots, n\}.$$

Eg 14.4: Consider (Ω, P) as above.

Let $\omega = (\omega_1, \dots, \omega_n) \in \Omega$. Permutations

$\omega_1 = 1 \rightarrow 1 \rightarrow \sigma^1 \in S_1$

$\omega_2 = 2 \rightarrow 1\ 2 \rightarrow \sigma^2 \in S_2$

$\omega_3 = 2 \rightarrow 1\ 3\ 2 \rightarrow \sigma^3 \in S_3$

$\omega_4 = 3 \rightarrow 1\ 3\ 4\ 2 \rightarrow \sigma^4 \in S_4$

$\vdots$

ω_n = position of n among existing $(n-1)$ numbers. — $\sigma^n \in S_n$.

Recursively, in σ^{i-1}, we put i in the ω_i^{th} position &

get σ^i. $\sigma^1 = 1$.

$$\Omega \ni \omega \longmapsto \sigma^n \in S_n$$

$\mathbb{P}(\sigma^n = \sigma) = ?$ for a fixed $\sigma \in S_n$.

(i.e., induced prob. on S_n)

observe $\omega \longmapsto \sigma^i \in S_i$ $i = 1, \ldots, n$.

TO PROVE THAT $\mathbb{P}(\sigma^i = \sigma) = \dfrac{1}{i!}$ $\forall\, i = 1, \ldots, n$ & $\sigma \in S_i$. — (1)

INDUCTION:

(1) holds for $i = 1$ trivially.

Assume (1) holds for $n-1$.

Let π be $\sigma \in S_n$ restricted to S_{n-1}. (remove n from σ).

i.e., if $n = 4$ & $\sigma = 1342$ then $\pi = 132$.

$$\mathbb{P}(\sigma^n = \sigma) = \mathbb{P}(\sigma^n = \sigma \mid \sigma^{n-1} = \pi)\, \mathbb{P}(\sigma^{n-1} = \pi)$$

Let $j \in [n]$ be the position on n in σ. (Above eg. position of 4 is 3 in σ)

$$= \mathbb{P}(w_n = j \mid \sigma^{n-1} = \pi)\, \mathbb{P}(\sigma^{n-1} = \pi)$$

σ^{n-1} depends on $(w_1, \ldots, w_{n-1})$. $\pi \in S_{n-1}$

$C = \{\sigma^{n-1} = \pi\} = B \times [n]$, $B \subseteq [1] \times [2] \times \cdots [n-1]$

$D = \{w_n = j\} = [1] \times [2] \times \cdots [n-1] \times A$ where $A = \{j\}$.

since our prob. space is prod. space, C & D are indep. [check]

So

$$\mathbb{P}(w_n = j \mid \sigma^{n-1} = \pi) = \mathbb{P}(D \mid C) = \mathbb{P}(D) = \frac{1}{n}.$$

So $\mathbb{P}(\sigma^n = \sigma) = \frac{1}{n} \mathbb{P}(\sigma^{n-1} = \pi) = \frac{1}{n!}$ (by induction step)

rand. permutation in S_{n-1}.

$\Rightarrow \sigma^n$ is a rand. permutation in S_n. ∎

LEMMA 14.4 If A, B, C are independent events then so are $A \cup B$ & C and $A \cap B$ & C.

Proof:
$$
\begin{aligned}
P((A \cup B) \cap C) &= P((A \cap C) \cup (B \cap C)) \\
&= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C) \quad (\text{IE}) \\
&= P(A)P(C) + P(B)P(C) - P(A \cap B)P(C) \quad (\text{indep.}) \\
&= P(C)\left(P(A) + P(B) - P(A \cap B)\right) \\
&= P(C)\,P(A \cup B) \quad (\text{IE})
\end{aligned}
$$

$\|^{ly}$ for $A \cap B$ & C.

Ex. 14.5. Generalize above to more events. For eg. A, B, C, D are indep. then so are $A \cup B$ & $C \cap D$.

Eg 14.6 There are n types of coupons. A coupon of type i is selected with prob. p_i $\left(p_i \geq 0 \text{ \& } \sum_{i=1}^{n} p_i = 1\right)$.

k indep. coupons are selected.

Find prob. $\exists$ at least one type i coupon & $\exists$ at least one type i or at least one type j.

Soln: $\Omega = [n]^k$; $p(\omega) = \prod_{l=1}^{k} p_{\omega_l}$ $\omega_l \in [n]$; $l = 1, \ldots, n$.

A_i = $\exists$ at least one type i coupon.

$\mathbb{P}(A_i) = 1 - \mathbb{P}(A_i^c) = 1 - \mathbb{P}(\text{no coupon of type } i)$

So $A_i^c = ([n] \setminus i)^k$ $\mathbb{P}(A_i^c) = (1-p_i)^k$ (Since A_i^c is a prod. set & using indep.)

$\mathbb{P}(A_i) = 1 - (1-p_i)^k$.

$\mathbb{P}(A_i \cup A_j) = \mathbb{P}(A_i) + \mathbb{P}(A_j) - \mathbb{P}(A_i \cap A_j)$

$= 1 - \mathbb{P}((A_i \cup A_j)^c) = 1 - \mathbb{P}(A_i^c \cap A_j^c)$

$A_i^c \cap A_j^c = ([n] \setminus \{i, j\})^k$ $\mathbb{P}(A_i^c \cap A_j^c) = (1 - p_i - p_j)^k$

$\mathbb{P}(A_i \cup A_j) = 1 - (1 - p_i - p_j)^k$ ∎

$\mathbb{P}(\bigcap_{i=1}^{n} A_i) = ?$ If $k < n$ $\mathbb{P}(\bigcap_{i=1}^{n} A_i) = 0$.

Eg 14.7 (Gambler's ruin problem).

Two gamblers A & B. Keep tossing a p-biased coin independently.
If H, B pays 1\$ to A & if T, A pays 1\$ to B.
Initially A has i \$ & B has $n-i$ \$.
Game over if A or B have 0 \$. A winner if A has n \$

Solution: p-biased coin $\mathbb{P}(\underset{1}{H}) = p$, $\mathbb{P}(\underset{0}{T}) = 1-p = q$

$$P_i = \underbrace{\mathbb{P}(A \text{ wins starting with } i \, \$)}_{A_i} = \mathbb{P}(A_i)$$

H_1 = event first toss is heads. T_1 = . . . is tails.

$$P_i = \mathbb{P}(A_i) \underset{(LTP)}{=} \mathbb{P}(A_i \mid H_1)\,\mathbb{P}(H_1) + \mathbb{P}(A_i \mid T_1)\,\mathbb{P}(T_1)$$

Given H_1, A has $i+1$ \$ & B has $n-i-1$ \$ & game continues if $i+1 \neq n$.

$i+1 \le n$

Since the next tosses are indep. of first toss, this is same as starting the game with $(i+1)\$$ for A.

So $\quad \mathbb{P}(A_i \mid H_1) = \mathbb{P}(A_{i+1}) = P_{i+1}$; $\mathbb{P}(A_i \mid T_1) = P_{i-1}$

$$\Rightarrow \quad P_i = p\,P_{i+1} + q\,P_{i-1}$$

$p+q=1$ $\quad \Rightarrow \quad p P_i + q P_i = p P_{i+1} + q P_{i-1}$

$$\Rightarrow \quad P_{i+1} - P_i = \frac{q}{p}\left(P_i - P_{i-1}\right)$$

$P_n = 1$. $\quad P_0 = 0$.

use this & $\quad P_i - P_{i-1} = \left(\frac{q}{p}\right)^{i-1} P_1$.

Add $P_i - P_{i-1}$
$+ P_{i-1} - P_{i-2}$
$+ \dots + P_2 - P_1$

$$\Rightarrow \quad P_i - P_1 = P_1\left[\left(\frac{q}{p}\right) + \dots + \left(\frac{q}{p}\right)^{i-1}\right]$$

$$1 = P_n = P_1\left[1 + \left(\frac{q}{p}\right) + \cdots + \left(\frac{q}{p}\right)^{n-1}\right]$$

$$= P_1 \frac{\left[1 - \left(q/p\right)^{n}\right]}{\left[1 - q/p\right]}, \quad \frac{q}{p} \neq 1.$$

$$\Rightarrow P_1 = \frac{1 - q/p}{1 - (q/p)^n} \Rightarrow P_i = \frac{1 - (q/p)^i}{1 - (q/p)^n}, \quad \frac{q}{p} \neq 1$$

If $\frac{q}{p} = 1$ ($\Leftrightarrow p = 1/2$) $\quad P_i = \frac{i}{n}$.

$Q_i = P(B \text{ winning starting with } n-i\ \$) = \dfrac{1 - (p/q)^{n-i}}{1 - (p/q)^n}$

Check $P_i + Q_i = 1$. $\forall i = 0, \dots, n$. $\blacksquare$