26|11 LECTURE 16 - RANDOM VARIABLES.

Not always do we observe outcome of an experiment but may be some 'observable' only.

For eg; experiment is 'n' coin tosses. We record/observe only # of heads.

# of heads is an 'observable'.

For eg; experiment is a roll of 2 dice. An observable is sum of the dice.

For eg; " is a shuffle of deck of 'n' cards.

# of matches is an observable.

Defn 16.1 $(\Omega, \mathbb{P})$ be a finite PS. A fn. $X: \Omega \longrightarrow \mathbb{R}$ is called a random variable (r.v.). We denote it by $X: (\Omega, \mathbb{P}) \longrightarrow \mathbb{R}$. on (Ω, P).

PROPN 16.2: Let $X: (\Omega, \mathbb{P}) \longrightarrow E \subset \subset \mathbb{R}$ be a r.v. $(E \subset\subset \mathbb{R}$, E is a finite subset of $\mathbb{R})$

(i.e. X takes values in E)

Define $p_X(x) = \mathbb{P}(\{X = x\}) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$, $x \in E$.

Then (E, p_X) is a PS i.e, p_X is a pmf on E.

Proof: (1) $p_X(x) \geq 0 \quad \forall x \in E$.

(2) $\{X = x\} := X^{-1}(x)$, $X^{-1}(x), x \in E$ are pairwise disjoint sets.

$$\sum_{x \in E} p_X(x) \underset{\text{defn of } p_X}{=} \sum_{x \in E} \mathbb{P}(X^{-1}(x)) \underset{\text{fin. add}}{=} \mathbb{P}\Big(\bigsqcup_{x \in E} X^{-1}(x)\Big) = \mathbb{P}(\Omega) = 1.$$

This proves p_X is a pmf on E. ∎

Defn 16.3 Given a r.v. $X : (\Omega, \mathbb{P}) \to \mathbb{R}$, we call the function $p_X : \mathbb{R} \to [0,1]$ defined as $p_X(x) := \mathbb{P}(X = x) = \mathbb{P}(X^{-1}(x))$, $x \in \mathbb{R}$ where $p_X(x) = 0$ if $x \notin X(\Omega)$. to be the $\boxed{\text{pmf of } X}$.

Define $\forall A \subseteq \mathbb{R}$, $\mathbb{P}_X(A) := \mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A))$

$[\{X \in A\} := X^{-1}(A)]$

$\mathbb{P}_X$ is called the prob. distribution of X.

Eg 1604.

(1) $\Omega = \{0,1\}$ $\quad \mathbb{P}(1) = p$.

$X : \Omega \to \mathbb{R}$ (identity fn.).
$\omega \mapsto \omega$.

$$p_X(1) = p, \quad p_X(0) = 1-p, \quad p_X(x) = 0 \quad \forall\, x \notin \Omega.$$

(2) $\Omega = \{0,1\}^n$ $\quad \mathbb{P}_p$ — product Bernoulli distribution. (n indep. biased coins)

$X : \Omega \to \mathbb{R}$
$\omega = (\omega_1, \dots, \omega_n) \mapsto \sum_{i=1}^{n} \omega_i$ $\quad$ (= # of heads in 'n' tosses)

$$p_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k = 0, \dots, n \\ 0 & k \notin \{0, \dots, n\} \end{cases}$$

$\mathbb{P}(X^{-1}(k)) = p_X(k)$

(3) $\Omega = [n]$, $\mathbb{P}$ — uniform prob. distribn.

$X : \Omega \to \mathbb{R}$ (identity)
$\omega \mapsto \omega$

$$p_X(x) = \begin{cases} \frac{1}{n} & \text{if } x \in [n] \\ 0 & \text{else.} \end{cases}$$

(4) $\Omega = S_n$, $\mathbb{P}$ — unif. Prob. distribution.

$$X : \Omega \to \mathbb{R}$$
$$\omega \longmapsto \sum_{i=1}^{n} \mathbb{1}[\omega_i = i] \quad (= \# \text{ of matches in a permutation})$$

$$p_X(x) = \begin{cases} \mathbb{P}(\exists \text{ exactly } `x' \text{ matches}) & \text{if } x \in [n] \\ 0 & x \notin [n] \end{cases}$$

$$\shortparallel$$

$$\mathbb{P}(X^{-1}(x))$$

Defn 16.3: Let $X : (\Omega, \mathbb{P}) \to \mathbb{R}$ be a r.v. Expectation of a r.v. X is defined as $\mathbb{E}[X] := \sum_{\omega \in \Omega} X(\omega) p(\omega)$.

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Suppose $X(\Omega) \subseteq E \subset\subset \mathbb{R}$ then

$$\mathbb{E}[X] = \sum_{\omega\in\Omega} X(\omega)\, p(\omega) = \sum_{x\in E} \sum_{\omega\in X^{-1}(x)} X(\omega)\, p(\omega) \quad \Big(\Omega = \bigsqcup_{x\in E} X^{-1}(x)\Big)$$

$$= \sum_{x\in E} x \sum_{\omega\in X^{-1}(x)} p(\omega)$$

$$= \sum_{x\in E} x\, \mathbb{P}(X^{-1}(x)) = \sum_{x\in E} x\, p_X(x) \quad -\ (1)$$

If $f : \mathbb{R} \to \mathbb{R} \ni f(x) \neq 0$ for only finitely many values x (i.e, $\mathbb{R} \setminus f^{-1}(0) \subset\subset \mathbb{R}$) then

$$\sum_x f(x) := \sum_{x\in\mathbb{R}} f(x) = \sum_{x : f(x)\neq 0} f(x).$$

Using this defn. & above observation, (1) we have that $\mathbb{E}[X] = \sum_x x\, p_X(x)$

as $P_X(x) = 0 \ \forall\, x \notin E$.

Eg 16.6 (Revisit Eg 16.4) (i) $X : \{0,1\} \to \mathbb{R}$ $\omega \mapsto \omega$. $p(1) = p$.

$$\mathbb{E}[X] = \sum_x x\, p_X(x) = 0\cdot(1-p) + 1\cdot p = p.$$

$\mathbb{E}[X]$ $\shortparallel$ Avg. of observable. p $\shortparallel$ $\mathbb{P}(X=1)$

(2) X = # of heads in 'n' indep. biased coin tosses. $\left(\Omega = \{0,1\}^n, \mathbb{P}_p \ldots\right)$

$$\mathbb{E}[X] = \sum_x x\, p_X(x)$$

Avg. # of heads in 'n' coin tosses.

$$= \sum_{k=0}^{n} k\, p_X(k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np. \text{ [check]}$$

(3) X = Position of a randomly picked card among 'n' cards.

$\left(\Omega = [n];\ \mathbb{P}\text{ - Unif.},\ X : \Omega \to \Omega \quad \omega \longmapsto \omega\right)$

$$\mathbb{E}[X] = \sum_{k=1}^{n} k\, P_X(k) = \sum_{k=1}^{n} \frac{k}{n} = \frac{n+1}{2}.$$

average position

(4) X = # of matches in a random permutation.

$$\mathbb{E}[X] = \sum_{k=0}^{n} k\, p_X(k) = \sum_{k=1}^{n} k\, \mathbb{P}[\exists \text{ exactly '}k\text{' matches}]$$

$$= \; ?? \quad \text{(check).}$$

(5) $X : (\Omega, \mathbb{P}) \to \mathbb{R}$, $A \subseteq \Omega$.

$$\omega \longmapsto \underline{\mathbb{1}}[\omega \in A].$$

[A is set of good outcomes
$X = \mathbb{1}[\text{good outcome}]$]

$$p_X(x) = \begin{cases} \mathbb{P}(A^c) & x = 0 \\ \mathbb{P}(A) & x = 1 \\ 0 & x \notin \{0,1\} \end{cases}$$

$$\| \quad \mathbb{P}(X^{-1}(x))$$

$$\| \quad \mathbb{P}(\{\omega : X(\omega) = x\})$$

$$\mathbb{E}[X] = \mathbb{P}(A).$$

[Every prob. of an event is $\mathbb{E}[X]$ for some X!]

$\boxed{\text{J.D.}}$ Take a prob. space $(\Omega, \mathbb{P})$, find an interesting event A & compute $\mathbb{P}(A)$.

$\boxed{\text{EJD}}$ " , define a R.V. & compute $\mathbb{E}[X]$.

Ex. 16.7

(1) Toss a fair coin. If heads I get 1 INR
If tails I lose 1 INR.

X = Net profit. $\mathbb{E}[X] = ?$

(2) Toss 'n' fair coins indep. Same as above for heads & tails.

X = Net profit after 'n' coin tosses. $\mathbb{E}[X] = ?$

(3) $\Omega = \{0, 1, 2\}$ $p(2) = 0$, $p(1) = p = 1 - p(0)$.

$X : \Omega \to \mathbb{R}$ identity fn. $\mathbb{E}[X] = ?$