26/11 LECTURE 17 - EXPECTATION

Defn 16.3 Given a r.v. $X : (\Omega, \mathbb{P}) \to \mathbb{R}$, we call the function $p_X : \mathbb{R} \to [0,1]$ defined as $p_X(x) := \mathbb{P}(X = x) = \mathbb{P}(X^{-1}(x))$, $x \in \mathbb{R}$ where $p_X(x) = 0$ if $x \notin X(\Omega)$. to be the $\boxed{\text{pmf of } X.}$

Define $\forall A \subseteq \mathbb{R}$, $\mathbb{P}_X(A) := \mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A))$ (Prob. distribn. of X.)
$[\{X \in A\} := X^{-1}(A)]$

$$\boxed{\mathbb{E}[X] := \sum_{\omega \in \Omega} X(\omega)\, p(\omega) = \sum_{x} x\, p_X(x).}$$

$\forall A \subseteq \Omega$, $X(\cdot) = \mathbb{1}_A(\cdot)$ then $\mathbb{E}[X] = \mathbb{P}(A)$

LEMMA 17.1 $X : (\Omega, \mathbb{P}) \to \mathbb{R}$ be a r.v. Let $f : \mathbb{R} \to \mathbb{R}$ be a fn. Then $f(X)$ is a r.v. & $\mathbb{E}[f(X)] = \sum_{x} f(x)\, p_X(x)$.

Proof: $f(X) : (\Omega, \mathbb{P}) \to \mathbb{R}$ is a composition of f & X & so a r.v.

$$\mathbb{E}[f(X)] = \sum_{\omega\in\Omega} f(X(\omega))\, p(\omega)$$

$$= \sum_{x\in E} \sum_{\omega\in X^{-1}(x)} f(x)\, p(\omega) \qquad (\text{let } E = X(\Omega))$$

$$= \sum_{x\in E} f(x)\, p_X(x) = \sum_{x} f(x)\, p_X(x). \qquad \blacksquare$$

Defn 17.2 Two r.v. X & Y are said to be $\boxed{\text{equal in distribution}}$ if $p_X = p_Y$ as fns in $\mathbb{R}$ i.e., $p_X(x) = p_Y(x)\ \forall x\in\mathbb{R}$.

Denoted by $X \overset{d}{=} Y$.

FACT: $X \overset{d}{=} Y \implies \mathbb{E}[f(X)] = \mathbb{E}[f(Y)]\ \forall f: \mathbb{R}\to\mathbb{R}$.

COR 17.3 Let $X: (\Omega, \mathbb{P}) \to \mathbb{R}$ be a r.v. & let $X(\Omega) \subseteq E \subset\subset \mathbb{R}$ hold.

Let $Y: (E, P_X) \to \mathbb{R}$ be identity fn.

Then $\mathbb{E}[f(X)] = \mathbb{E}[f(Y)] \quad \forall f : \mathbb{R} \to \mathbb{R}$.

Proof: Exercise. $\blacksquare$

For eg. Choose $E = \{x \in \mathbb{R} : p_X(x) > 0\} \subset\subset \mathbb{R}$

$\hookrightarrow$ Ω is finite.

Terminology: (1) X is Bernoulli(p) r.v. if $p_X(0) = 1 - p = 1 - p_X(1)$.

(2) X is Binomial(n, p) r.v. if $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$

(3) X is Hypergeometric (n, m, r) r.v. if $\mathbb{P}(X = k) = \dfrac{\binom{m}{k}\binom{n-m}{r-k}}{\binom{n}{r}}$

$X \overset{d}{=} \text{Ber}(p) \Rightarrow X$ is Bernoulli(p) r.v.

$X \overset{d}{=} \text{Bin}(n,p) \Rightarrow X$ . . . .

$X \overset{d}{=} \text{HyperG}(n,m,r) \Rightarrow$ . . .

$X \overset{d}{=} \text{Unif}([n]) \Rightarrow \mathbb{P}(X=k) = \dfrac{1}{n}, \; k = 1, \ldots, n$.

PROPOSITION: (1) X_1, X_2 are r.v. then $\mathbb{E}[aX_1 + bX_2] = a\mathbb{E}[X_1] + b\mathbb{E}[X_2]$
17.4
(Linearity of Expectation) $\forall\, a, b \in \mathbb{R}$.

(2) X_1, X_2 are r.v. $\ni$ $P_{X_1} \le P_{X_2}$ then $\mathbb{E}[X_1] \le \mathbb{E}[X_2]$.

(3) Let $X_1, X_2 : (\Omega, \mathbb{P}) \longrightarrow \mathbb{R}$ be r.v. $\ni$ $X_1(\omega) \le X_2(\omega)$ (Monotonicity of $\mathbb{E}$)
then $\mathbb{E}[X_1] \le \mathbb{E}[X_2]$.

Proof: (1) Assume $X_1, X_2 : (\Omega, \mathbb{P}) \longrightarrow \mathbb{R}$ [Think what happens if not?]
then $aX_1 + bX_2$ is also a r.v.
By defn
$$\mathbb{E}[aX_1 + bX_2] \underset{\text{(defn)}}{=} \sum_{\omega \in \Omega} (aX_1(\omega) + bX_2(\omega))\, p(\omega)$$
$$\underset{\text{(split the sum as it is fin.)}}{=} \sum_{\omega \in \Omega} aX_1(\omega)\, p(\omega) + \sum b X_2(\omega)\, p(\omega)$$
$$\underset{\text{(defn)}}{=} a\mathbb{E}[X_1] + b\mathbb{E}[X_2]$$

(2) $\mathbb{E}[X_1] = \sum_x x\, p_{X_1}(x) \underset{\text{assumption}}{\le} \sum_x x\, p_{X_2}(x) = \mathbb{E}[X_2]$

(3) $\mathbb{E}[X_1] = \sum_\omega X_1(\omega)\, \mathbb{P}(\omega) \le \sum_\omega X_2(\omega)\, \mathbb{P}(\omega) = \mathbb{E}[X_2]$.

Eg 17.5 Let $(\Omega, \mathbb{P})$ be a PS. & $A_1, \ldots, A_n$ be events.

Define $X = \mathbb{1}_{A_1 \cup \cdots \cup A_n}$ $\left(\omega \longmapsto \mathbb{1}[\omega \in A_1 \cup \cdots \cup A_n]\right)$

$X_i = \mathbb{1}_{A_i}$ $i = 1, \ldots, n$.

Trivially $X \le \sum_{i=1}^n X_i$.

By mon. & linearity $\mathbb{E}[X] \le \sum_{i=1}^n \mathbb{E}[X_i]$

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) \le \sum_{i=1}^n \mathbb{P}(A_i) \quad \text{[union bound]}$$

(where $\mathbb{E}[X] = \mathbb{P}(A_1 \cup \cdots \cup A_n)$ and $\mathbb{E}[X_i] = \mathbb{P}(A_i)$)

Ex. 17.6. Try proving I-E & Bonferroni ineq. using lin. & mon.

Eg: $X \overset{d}{=} \text{Bin}(n,p)$, $\mathbb{E}[X] = \sum_{k=0}^{n} k\, p_X(k)$
(7.7)

Let $\Omega = \{0,1\}^n$, $\mathbb{P}_p$ – prod. Bernoulli distribution.

$Y_i : \Omega \to \mathbb{R}$; $\omega \longmapsto \omega_i$. (Heads or tails in the i^{th} toss)
$\omega = (\omega_1, \dots, \omega_n)$

$p_{Y_i}(0) = 1 - p = 1 - p_{Y_i}(1)$. $\Rightarrow \mathbb{E}[Y_i] = p$.
$p_{Y_i}(0) = \mathbb{P}(Y_i = 0)$

$Y = Y_1 + \cdots + Y_n$ (= # heads in n tosses)

<u>Check</u> $Y \overset{d}{=} X$ $\left(p_Y(k) = p_X(k) \quad \forall k = 0, \dots, n \right)$

So by our fact $\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[Y_1 + \cdots + Y_n]$
$= \sum_{i=1}^{n} \mathbb{E}[Y_i] = np$. ∎
(linearity)

Eg 17.8.

X = # matches in a random permutation.

$X : S_n \to \mathbb{R}$ $\quad X(w) = \sum_{i=1}^{n} \mathbb{1}[w(i) = i]$

$\mathbb{P}$ – U.P.D.

$$\mathbb{E}[X] = \sum_{k=1}^{n} k \, p_X(k).$$

Define $X_i : S_n \to \mathbb{R} \ni X_i(w) = \mathbb{1}[w(i) = i]$

$$\mathbb{E}[X_i] = \mathbb{P}[X_i(w) = 1] = \mathbb{P}(\{w : w(i) = i\})$$

$$= \frac{1}{n} \text{. (prob. } i^{th} \text{ co-ordinate is fixed)}$$

By linearity $\mathbb{E}[X] = \mathbb{E}[X_1 + \cdots + X_n]$ $\quad (X = X_1 + \cdots + X_n)$.

$$= n \cdot \frac{1}{n} = 1.$$

$\mathbb{E}$(# matches$]$ $= 1$.

Ex. Work out # of empty bins in FD, MB, BE distri

$\mathbb{E}[X^2]$, $\mathbb{E}[X^3]$, ..