28/9 LECTURE - 2.

Basics of set theory & Combinatorial analysis. - Refer to D'Angelo & West; Ross - Ch. 1 & parts of Ch. 2; K. L. Chung - Ch. 1. ("Elementary Prob.")

(TA will revise some on Friday).

Say we perform an "experiment". Assume there are finitely many possible outcomes: Our goal in probability to mathematically model the "experiment" and make "prediction" about outcomes.
↳ (indicates there is uncertainity about outcomes).

The bigger focus here - How to "understand" the model?

(Finite) SAMPLE SPACE: Ω - a finite set ["set of all possible outcomes" for the experiment]
(∞ later)

Event: An event is a subset of Ω. ["collection of outcomes"]

Denote events by A, B, C & so on.

Notation: In general small letters (say x, y, w, w_1, x_1 & so on) for elements of Ω & capitals for subsets or events.

Eg. 1: $\Omega = \{0, 1\}$ [Toss a coin; H = 1 & T = 0 → possible outcomes]
EXPERIMENT
$A = \phi, \{0\}, \{1\}, \Omega$. Play a game; W = 1 & L = 0]

Eg 2: $\Omega = \{1, 2, \ldots, 6\}$ [Roll a dice - 6 possible outcomes - $1, \ldots, 6$]

Eg. A = Set of all even no.s $= \{2, 4, 6\}$.

Eg. 3: $\Omega = \{0, 1\}^n$; $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ - natural numbers.
$\{(a_1, \ldots, a_n) : a_i \in \{0, 1\}\}$ [Toss a coin 'n' times; Play a game of 'n' rounds]

Ex: Compute cardinalities of sample spaces & some events
($|\Omega|$) ($|A|$)
└ cardinality.

Eg. 4: $\Omega = \{0, 1, \ldots, n\}$ - [What is experiment?]

Eg 5: $\Omega = [52] = \{1, \ldots, 52\}$ $\quad ([n] := \{1, 2, \ldots, n\})$

(Selecting a card from a pack)

A – set of spades. $\quad |A| = ?$

[label cards by $1, \ldots, 52$]

Convention: $\underbrace{\{1, \ldots, 13\}}_{\text{spade}}$ $\underbrace{\{14, \ldots, 26\}}_{\text{clove}}$, $\underbrace{\{27, \ldots, 39\}}_{\diamondsuit}$, $\underbrace{\{40, \ldots, 52\}}_{\text{Hearts}}$.

Examples from statistical physics.

Particles – Electrons / Protons / Neutrons.

Particles occupy cells / sites. $\longrightarrow$ always labelled.

Assume there are 'n' cells & 'r' particles. $n, r \in \mathbb{N}$.

Two possible cases – labelled particles (i.e., we can distinguish particles)

– unlabelled – particles (we know no: of particles in a cell)

Eg 6: Ω = possible positions of labelled particles

$= \{(p_1, \ldots, p_r) : 1 \leq p_i \leq n$ for $1 \leq i \leq r\}$ $p_i \in \mathbb{N}$ — such that or a constraint

(p_i – position of the ith particle).

$= \{1, \ldots, n\}^r =: [n]^r$ $\longrightarrow$ [defn. of a "quantity"]

A = At least one cell is empty

A_i = ith cell is empty $= \{(p_1, \ldots, p_r) \in \Omega : p_k \neq i \ \forall\, 1 \leq k \leq r\}$ — belongs to; for all; such that

$(\subseteq \Omega)$

So $A = \bigcup_{i=1}^{n} A_i \subseteq \Omega$ ('cos $A_i \subseteq \Omega$)

if $n > r$ then $A = \Omega$. [Ex.]

Ex: Describe Ω when two lab. particles can't occupy the same cell. (Ω here is a subset of $[n]^r$)

(n cells & r particles)

Ex: Cells labelled $1, \ldots, n$ — r Particles cannot occupy successive cells.
(spectrum frequencies) (Successive frequencies can't be allocated)

Unlabelled Particles [See Feller I.2 for other contexts]

Eg: $\Omega_{r,n}$ = possible configurations of unlabelled particles

$$= \{ (r_1, \ldots, r_n) : r_i \in \{0, \ldots, r\} \ \forall \ 1 \le i \le n \ \& \ r_1 + \ldots + r_n = r. \}$$

r_i = # of particles in cell i.

interesting events A = ?

Eg: $\Omega^*_{r,n}$ = possible config of unlab. particles such that no two particles at same site.

$$= \{ (r_1, \ldots, r_n) : r_i \in \{0,1\} \ \forall \ 1 \le i \le n \ \& \ r_1 + \ldots + r_n = r \}$$

[Note: $\Omega^*_{r,n} \subseteq \Omega_{r,n}$; $\Omega^*_{r,n} \subseteq \{0,1\}^n$]

Experiment is "interesting" if only outcomes "can differ" every time you perform the experiment.

Natural to consider "the probability" of an outcome.

(means " certainity with which outcome occurs").

We need to define $\mathbb{P}(\omega) \ \forall \ \omega \in \Omega$.

Von Mises (Frequentist approach) : $\mathbb{P}(\omega) \approx$ "proportion' of 'occurence of ω when experiment is repeated many times!

Problems: Experiment has to be repeated many times — say for 'n' large.

Also say you toss a coin 1000 times, 60% heads $\Rightarrow$ "$\mathbb{P}(1) = 0.6$"

I toss the coin further 9000 times, total 55% heads $\Rightarrow$ "$\mathbb{P}(1) = 0.55$"

number

$$\frac{\text{\# of occurence of } \omega \text{ in 'n' experiments}}{n} \longrightarrow ?? \quad \text{as } n \to \infty$$

Defn: Let Ω be a finite set. Let $p : \Omega \to [0,1]$ be a function such that (s.t.) $\sum_{\omega \in \Omega} p(\omega) = 1$. Then (Ω, p) is a finite prob. space.

p is called prob. mass function (pmf).

Define $\mathbb{P} : 2^{\Omega} \to [0,1]$ as $\mathbb{P}(A) = \sum_{\omega \in A} p(\omega)$, $A \subseteq \Omega$ (subset of Ω)

2^{Ω} = Power-set of Ω = $\{A : A \subseteq \Omega\}$

Then $\mathbb{P}$ is called prob. distribution.

[Note: $|2^{\Omega}| = 2^{|\Omega|}$]

Find interesting (Ω, p) & an event A and compute $\mathbb{P}(A)$.