04/12 LECTURE - 20. DISCRETE PROB. SPACES.

Suppose Ω is a countably infinite sample space. [Ω - discrete if Ω is count.]

Can we define prob. (i.e., pmf) on it?

Ω is countable if $\exists$ a bijection from Ω to $A \subseteq \mathbb{N}$.

(or $\exists$ an injection from Ω to $\mathbb{N}$).

If p is pmf on Ω then $p(\omega) \geq 0 \;\; \forall \omega \in \Omega$

& $\sum_{\omega \in \Omega} p(\omega) = 1$.

For eg. $\Omega = \mathbb{N}$. Need to define $p : \mathbb{N} \to [0,1] \ni \sum_n p(n) = 1$.

Suppose $a_n := p(n)$. We want a seq. $a_n \geq 0 \ni \sum_{n=1}^{\infty} a_n = 1$.

More so, we only need $b_n \geq 0 \ni \sum_{n=1}^{\infty} b_n < \infty$.

If such a b_n exists set $a_n = \frac{b_n}{b}$ where $b = \sum_{n=1}^{\infty} b_n$.

So $a_n \geq 0$ & $\sum_{n=1}^{\infty} a_n = 1$. Thus $p(n) = a_n$ is a valid pmf.

When we write $\sum_{n=1}^{\infty} a_n$, there is an implicit order.

For any seq (a_n) $\sum_{n=1}^{\infty} a_n := \lim_{m \to \infty} \sum_{n=1}^{m} a_n$ (if it exists)

Fact: (1) If $a_n \geq 0$. then $\sum_{n=1}^{\infty} a_n = \infty$ or $\sum_{n=1}^{\infty} a_n < \infty$.

(2) Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection, $a_n \geq 0$ & $\sum_{n=1}^{\infty} a_n < \infty$.

then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}$. — (1)

In this case we set $\sum_n a_n = \sum_{n=1}^{\infty} a_n$.

(3) Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection; a_n any sequence such that $\sum_{n=1}^{\infty} |a_n| < \infty$. Then (1) holds

& $\sum_n a_n = \sum_{n=1}^{\infty} a_n$.

(4) Given a seq. a_n, define $a_n^+ := \max\{a_n, 0\} \geq 0$

$a_n^- := \max\{-a_n, 0\} \geq 0$.

observe $a_n = a_n^+ - a_n^-$; $|a_n| = a_n^+ + a_n^-$.

& $a_n^+ a_n^- = 0$.

Check: (1) $\sum_{n=1}^{\infty} |a_n| < \infty$ iff $\sum_{n=1}^{\infty} a_n^+ < \infty$ & $\sum_{n=1}^{\infty} a_n^- < \infty$.

(2) If $\sum_{n=1}^{\infty} |a_n| < \infty$ then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-$.

Defn 20.1 Let Ω be countable. Let $f: \Omega \to \mathbb{R}$. We say that $\sum_{\omega \in \Omega} f(\omega) = \sum_{\omega} f(\omega)$ is well-defined if

$$\sum_{\omega \in \Omega} |f(\omega)| := \sum_{n=1}^{\infty} |f(\sigma^{-1}(n))| < \infty$$

where $\sigma: \Omega \to \mathbb{N}$ is an injection

& set $\sum_{\omega} f(\omega) = \sum_{n=1}^{\infty} f(\sigma^{-1}(n))$. [$\sigma^{-1}(n)$ isn't defined $f(\sigma^{-1}(n)) = 0$]

From above facts the well-defined-ness is indep. of σ.

Defn 20.2 Given a sequence $\{a_n ; n \geq 0\}$ we say than $\sum_n a_n$ is well-defined if $\sum_{n=1}^{\infty} |a_n| < \infty$

& we set $\sum_n a_n := \sum_{n=1}^{\infty} a_n$.

Defn 20.3 p is a pmf on $\mathbb{N}^*$ if $p(n) \geq 0 \ \forall n \geq 0$ & $\sum_n p(n) = 1$.

$(\mathbb{N}^* := \mathbb{N} \cup \{0\})$ (i.e., $\sum_n p(n)$ is well-defined)

Eg 20.4 If $\Omega \subset\subset \mathbb{N}^*$ & (Ω, p) is a P.S. then p is a pmf on $\mathbb{N}^*$ (set $p(n) = 0 \ \forall n \notin \Omega$).

Ex 20.5 (1) $p(n) = \frac{1}{2^n}$; $n \geq 1$ $p(0) = 0$.

(2) $p(n) = p(1-p)^{n-1}$, $n \geq 1$ $p \in (0, 1]$

(3) $p(n) = p(1-p)^n$; $n \geq 0$. $p \in (0,1]$

(4) $p(n) = \dfrac{e^{-\lambda} \lambda^n}{n!}$; $n \geq 0$ $\lambda \geq 0$. $(0^0 = 1)$.

Check all of the above & below are pmf on $\mathbb{N}^*$.

(5) $p(n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}$ $n \geq 1$; $p \in (0,1]$, $r \geq 1$.

$\boxed{\text{EXTRA EXTRA:}}$ If $f : \Omega \to \mathbb{R}_+ = [0, \infty)$

then define $\sum_{\omega} f(\omega) := \sup \left\{ \sum_{\omega \in A} f(\omega) : A \subset\subset \Omega \right\}$.

Show that $\sum_{\omega} f(\omega) < \infty$ iff $\sum_{\omega} f(\omega) < \infty$ as in Defn 20.1.
(i.e., via injection to $\mathbb{N}$).

For more see Ch-5 of Manjunath Krishnapur's notes.