9/12 LECTURE 21 — DISCRETE PD.

p is a pmf on $\mathbb{N}^*$ if $p(n) \geq 0$ & $\sum_n p(n) = 1$.

[$(\mathbb{N}^*, p)$ is a PS.]

Define $\mathbb{P}(A) := \sum_{w \in A} p(w)$ $A \subseteq \mathbb{N}^*$. (1)

Ex. 21.1 Check (i) $\mathbb{P}(A) \geq 0$ $\forall A \subseteq \mathbb{N}^*$

(ii) $\mathbb{P}(\mathbb{N}^*) = 1$.

(iii) $\mathbb{P}(A_1 \sqcup \cdots \sqcup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i)$ (finite add.)

($A_1, \ldots, A_n$ can also be ∞ sets)

$\mathbb{P}$ defined as in (1) is called PD. (prob. distribution).

Qn: Suppose $\mathbb{P}$ satisfies (i), (ii) & (iii) then is there a pmf $p : \mathbb{N}^* \to [0,1] \ni \mathbb{P}(A) = \sum_{w \in A} p(w)$?

[We know ans is Yes if $\mathbb{P}$ corresponds to a finite prob. space i.e., $\mathbb{P}(\Omega) = 1$ for some $\Omega \subset\subset \mathbb{N}^*$]

Approach to solution: Define $p(w) = \mathbb{P}(\{w\})$.

Easily $p(w) \geq 0$.

Remains to check $\sum_w p(w) = 1. \Leftrightarrow \sum_{w \in \mathbb{N}^*} \mathbb{P}(\{w\}) = 1$

(iii) $\Rightarrow \sum_{w=1}^{n} \mathbb{P}(\{w\}) = \mathbb{P}(\{1, \ldots, n\})$ $\forall n \geq 0$.

Not enough to show $\sum_{w \in \mathbb{N}^*} \mathbb{P}(\{w\}) = 1$. X.

Defn 21.2 A fn $\mathbb{P} : 2^{\mathbb{N}^*} \longrightarrow \mathbb{R}$ is called a PD if

(i) $\mathbb{P}(A) \geq 0$ $\forall A \subseteq \mathbb{N}^*$

(ii) $\mathbb{P}(\mathbb{N}^*) = 1$

(iii) $\mathbb{P}\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$ (Count. add.)

[A_i's are pairwise disjoint].

Ex 2.1.3 Check if $\mathbb{P}$ is a PD on $\mathbb{N}^*$ then $p(\omega) = \mathbb{P}(\{\omega\})$ is a pmf on $\mathbb{N}^*$.

Also check count add. $\Rightarrow$ fin. add.

PROPN 2.1.4 Let $(\mathbb{N}^*, \mathbb{P})$ on $(\mathbb{N}^*, p)$ be a PS.

(1) $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$ (count. subadd; union bd; Boole's ineq)

(2) $\mathbb{P}(A) \subseteq \mathbb{P}(B)$ $\forall A \subseteq B \subseteq \mathbb{N}^*$ (Mon.)

(3) $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$ $\forall A \subseteq B \subseteq \mathbb{N}^*$

(4) I-E holds for $\mathbb{P}(A_1 \cup \cdots \cup A_n)$ $\forall n \geq 1$

(5) Bonferroni ineq. hold for $\mathbb{P}(A_1 \cup \cdots \cup A_n)$ $\forall n \geq 1$.

Proof: Ex!! If the statement involves only "finitely many" sets, same proof as for fin. prob. spaces.

Else try using count. add. ■

LEMMA 2.1.5 Let $A_n \uparrow A$ i.e., $A_n \subseteq A_{n+1}$ & $\bigcup_{n=1}^{\infty} A_n = A$. ($A_n$ increases to A)

Then $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.

Holds also for $A_n \downarrow A$ i.e., $A_n \supseteq A_{n+1}$ & $\bigcap_{n=1}^{\infty} A_n = A$.

Proof: Assume $A_n \uparrow A$. $\Rightarrow$ $A_n \subseteq A_{n+1}$ & $A_n \subseteq A$

& so $\mathbb{P}(A_n) \leq \mathbb{P}(A_{n+1}) \leq \mathbb{P}(A)$ (monotonicity of $\mathbb{P}$)

$\mathbb{P}(A_n)$ is an increasing seq. & bdd by $\mathbb{P}(A)$.

$\Rightarrow$ $\lim_{n\to\infty} \mathbb{P}(A_n)$ exists by MCT (monotone convergence thm)

Ex: show that $\sup_{n\geq 1} \mathbb{P}(A_n) = \mathbb{P}(A)$. [use count. add.]

or show $\mathbb{P}(A \mid A_n) \longrightarrow 0$ as $n \to \infty$. ["]

Suppose $A_n \downarrow A$. Then $A_n^c \uparrow A^c$.

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} 1 - \mathbb{P}(A_n^c)$$

$$= 1 - \lim_{n\to\infty} \mathbb{P}(A_n^c) = 1 - \mathbb{P}(A^c) = \mathbb{P}(A)$$

$(A_n^c \uparrow A^c)$. ∎

RMK 21.5 Indep. of events, Condnl. prob. are all defined as before i.e.,

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \quad \text{if } \mathbb{P}(A) > 0.$$

Events $A_1, \dots, A_n$ are indep. if

$$\mathbb{P}\Big(\bigcap_{j\in J} A_j\Big) = \prod_{j\in J} \mathbb{P}(A_j) \quad \forall J \subseteq [n].$$

Defn 21.6 X is a discrete r.v. (in $\mathbb{N}^*$) if $\mathbb{P}(X \in \mathbb{N}^*) = 1$.

i.e., X takes values only in $\mathbb{N}^*$.

EQUIVALENTLY, $\exists$ a fn $P_X : \mathbb{R} \to [0,1] \ni \sum_{n \in \mathbb{N}^*} P_X(n) = 1$.

& interpreted as $P_X(n) = \mathbb{P}(X = n)$.

P_X is called as pmf of X.

Define $\mathbb{P}(X \in A) = \sum_{x \in A} P_X(x) = \sum_{x \in A \cap \mathbb{N}^*} P_X(x)$

(as $P_X(x) = 0$ if $x \notin \mathbb{N}^*$)

Again when we say X is a r.v., we are also given a pmf P_X!

Eg 21.7 A p-biased coin is tossed repeatedly & independently until we get heads. ($p \in (0,1]$)

Let $X = \#$ of tosses.

$p_X(n) = P(X=n) = P(\text{first } n-1 \text{ tosses are tails \& } n^{th} \text{ toss is heads})$

$= (1-p)^{n-1}\, p$ (the tosses are indep. & prob. heads $= p$)

Check $p_X(n) \geq 0 \;\forall\, n \geq 1$ & $\sum_n p_X(n) = 1$.

X with pmf $p_X(n) = (1-p)^{n-1} p,\ n \geq 1$ is called GEOMETRIC RV. (Geom(p)).

Check $\sum_n p_X(n) = \sum_n P(n-1 \text{ tails \& } n^{th} \text{ heads})$

(count. add) $= P\left(\bigcup_n \{n-1 \text{ tails \& } n^{th} \text{ heads}\}\right)$

$= P(\text{eventually there is an head})$

$P(X \geq n) = \sum_{k \geq n} p_X(k) = \ldots$

$P(X \geq n) = P(\text{first } n-1 \text{ tosses are tails}) = (1-p)^{n-1},\ n \geq 1.$

Eg 21.8 A p-biased coin is tossed repeatedly & indep. until we get 'r' heads. $(p \in (0,1])$

$X = \#$ of tosses

$p_X(n) = P(X=n)$

$= P(r^{th} \text{ head in } n^{th} \text{ toss})$

$= P(r-1 \text{ heads in } n-1 \text{ tosses \& } n^{th} \text{ toss is a head})$

(since tosses are indep.) $= P(r-1 \text{ heads in } n-1 \text{ tosses})\, P(n^{th} \text{ toss is a head})$

$= \binom{n-1}{r-1} p^{r-1} (1-p)^{n-r} \times p$ → Binomial prob.

$= \binom{n-1}{r-1} p^{r} (1-p)^{n-r}$

Check $\sum_{n \geq 1} p_X(n) = 1$. (X, p_X) is a NEG. BIN. RV [NBin (r, p)]

$NBin(1,p) = Geom(p)$:

Eg 21.9 Let X be a $Geom(p)$ r.v. i.e., $p_X(n) = (1-p)^{n-1} p \;\; \forall\, n \geq 1$.

$$\mathbb{P}(X > k+l \mid X > k) = ?$$

$\{X > k\} = \{\text{no heads until } k\}$

$$\mathbb{P}(X > k+l \mid X > k) = \mathbb{P}(\text{no heads until } k+l \mid \text{no heads until } k)$$

$$\mathbb{P}(X > k+l \mid X > k) \underset{(\text{defn. of CP})}{=} \frac{\mathbb{P}(\{X > k+l\} \cap \{X > k\})}{\mathbb{P}(X > k)}$$

$$= \frac{\mathbb{P}(X > k+l)}{\mathbb{P}(X > k)} \underset{Geom(p)}{=} \frac{(1-p)^{k+l}}{(1-p)^k} = (1-p)^l$$

$\Rightarrow \mathbb{P}(X > k+l \mid X > k) = \mathbb{P}(X > l)$ [Memorylessness of $Geom(p)$ r.v.]

$\Leftrightarrow \mathbb{P}(X > k+l) = \mathbb{P}(X > k)\,\mathbb{P}(X > l)$

Eg 21.10 A p-biased coin is tossed repeatedly & indep. until we get (0,1) or (1,0), T,H or H,T.

If we stop with (T,H), set $Y = 0$.

(H,T), set $Y = 1$. [Y is a new r.v.]

What is $p_Y(\cdot) = ?$

$$p_Y(0) = \mathbb{P}(\text{end with } (T,H)) \to B$$

$$\underset{(\text{count. add.})}{=} \sum_{n=2}^{\infty} \mathbb{P}(\text{we end at } n \text{ with } (T,H)) \to B_n \quad \& \; B = \bigcup_{n=2}^{\infty} B_n$$

$$\underset{(\text{indep.})}{=} \sum_{n=2}^{\infty} \mathbb{P}(n-1 \text{ Tails } \& \; n^{th} \text{ heads}) = p \sum_{n=2}^{\infty} (1-p)^{n-1}$$

$$= (1-p).$$

$X = \#$ of tosses; Compute $p_X(n)$.