10/12 LECTURE 22. DISRETE RV & EXPECTATIONS

Eg 22.1 I toss two indep. coins (p-biased) repeatedly until I get HT or TH.

coin 1, coin 2 ; coin 1 & coin 2

Two coins have same value, we toss again.
If they have diff. value, we stop.

$Y = 0$ if HT

$Y = 1$ if TH.

$P_Y(0) = P(\text{we end with HT}) \to B$ $\quad B = \bigsqcup_{n=1}^{\infty} B_n$

(count. add.) $= \sum_{n=1}^{\infty} P(\text{HH or TT in the first } n-1 \text{ tosses \& HT in the } n^{th} \text{ toss}) \to B_n$

$= \sum_{n=1}^{\infty} P(\text{HH or TT})^{n-1} P(\text{HT})$ (by indep. of tosses)

(by indep.) $= \sum_{n=1}^{\infty} (1-2pq)^{n-1} pq$ $\quad q = 1-p$. $\quad (p^2+q^2+2pq = 1)$

$P(\text{HT}) = P(\text{coin 1 is H \& coin 2 is T}) = pq$ (indep.)

$P(\text{HH or TT in } n+ \text{ tosses}) = P(\text{coin 1} = \text{coin 2 in the first } n-1 \text{ tosses})$

So $P_Y(0) = \dfrac{pq}{2pq} = \dfrac{1}{2}$.

So $P_Y(1) = \dfrac{1}{2} = 1 - P_Y(0)$.

$Y \overset{d}{=} \text{Unif}\{0,1\} \overset{d}{=} \text{Ber}(1/2)$ (unbiased coin!!)

Ex* Using p-biased coin, can you define $\text{Unif}\{1, \dots, k\}$ r.v.?

$X = \#$ tosses before HT or TH

$P_X(m) = 2pq(1-2pq)^{m-1}$. $\quad X \overset{d}{=} \text{Geom}(2pq)$.

Defn 22.2. Let (X, P_X) be a discrete r.v with values in $\mathbb{N}^*$.
Let $f : \mathbb{N}^* \to \mathbb{R}$.

Define $\mathbb{E}[f(X)] := \sum_{x \in \mathbb{N}^*} f(x) P_X(x)$ if $\sum_{x \in \mathbb{N}^*} |f(x)| P_X(x) < \infty$

[Exp. of $f(X)$ or mean of $f(X)$ or Avg. of $f(X)$]

else $\mathbb{E}[f(X)]$ is undefined.

[Recall if $\sum_{x \in \mathbb{N}^*} |g(x)| < \infty$ then $\sum_{x\in\mathbb{N}^*} g(x)$ is well-defined
$\qquad\qquad\quad \overset{\shortparallel}{\sum_{n=0}^{\infty} |g(n)|}$ & $\sum_{x \in \mathbb{N}^*} g(x) = \sum_{n=0}^{\infty} g(n)$.]

Again given discrete r.v. X, important to compute
$\mathbb{E}[X]$ - mean, $VAR[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ - VARIANCE
k^{th} moment of X - $\mathbb{E}[X^k]$

we can describe a r.v. via its moments!!!

So $\mathbb{E}[X]$ is well-defined if $\sum_x |x| p_X(x) = \sum_x x\, p_X(x) = \sum_{n=0}^{\infty} n p_X(n) < \infty$.
$VAR[X]$ is . . if $\sum_x x^2 p_X(x) = \sum_{n=0}^{\infty} n^2 p_X(n) < \infty$

So if $\mathbb{E}[X^k]$ is well-defined if $\sum_{n=1}^{\infty} n^k p_X(n) < \infty$.

Ex 22.3: If $\mathbb{E}[X^m]$ is well-defined for some $m \in \mathbb{N}$, (X is $\mathbb{N}^*$-valued)
show that $\mathbb{E}[X^k]$ is well-defined $\forall\, k \le m$.

Eg 22.4: $X \overset{d}{=} Geom(p)$. $p_X(k) = p(1-p)^{k-1}$ $k \geq 1$. $p > 0$
$\mathbb{E}[X]$ is well-defined if $\sum_{n=1}^{\infty} n\, p_X(n) < \infty$

$$\sum_{n=1}^{\infty} n\, p_X(n) = p \sum_{n=1}^{\infty} n (1-p)^{n-1} \underset{\text{check}}{=} \frac{1}{p} < \infty.$$

So $\mathbb{E}[X]$ is well-defined & $\mathbb{E}[X] = \sum_{n=1}^{\infty} n\, p_X(n) = \frac{1}{p}$.

$$\sum_{n=1}^{\infty} n(1-p)^{n-1} = -\frac{d}{dp}\left(\sum_{n=0}^{\infty}(1-p)^n\right)$$

Compute $VAR[X]$, $\mathbb{E}[X^k]$, $k \geq 2$

Eg 22.5: $X \overset{d}{=} NBin(r,p)$ $\qquad p_X(m) = \binom{m-1}{r-1} p^r (1-p)^{m-r}$
$\shortparallel$
{# of tosses until r heads}

$$\sum_{n=1}^{\infty} n \binom{n-1}{r-1} p^r (1-p)^{n-r} = p^r \sum_{n=1}^{\infty} n \binom{n-1}{r-1} (1-p)^{n-r}$$

$$= r p^r \sum_{n=1}^{\infty} \binom{n}{r} (1-p)^{n-r}$$

$$= \frac{r}{p} \sum_{n=1}^{\infty} \underbrace{\binom{n}{r} p^{r+1} (1-p)^{n-r}}_{\text{pmf of another NBin}}$$

$$= \frac{r}{p} < \infty.$$

And so $\mathbb{E}[X] = \frac{r}{p}$ (Avg. waiting for r heads

= Avg. waiting for 1^{st} head + . . .

+ . . . r^{th} head)

VAR[X], $\mathbb{E}[X^k] = ?$ $k \geq 2$.

Eg 22.6. Let us take a queue. (Q at a bus-stop, railway station or internet servers)

Customers arrive in the Q at rate λ, $\lambda \in (0, \infty)$

(1) Avg. # of customers arriving btw $[t, t+s] = \lambda s$

(2) No two customers arrive close to each other.

$\hookrightarrow$ # of customers arriving in $[t, t+\delta] \in \{0, 1\}$ if δ is very small

(3) Customers arrive indep.

i.e., # in $[a,b]$ & # in $[c,d]$ are indep. if $[a,b] \cap [c,d] = \phi$

$\mathbb{P}(\# \text{ in } [a,b] = k, \# \text{ in } [c,d] = j) = \mathbb{P}(\quad = k)\ \mathbb{P}(\quad = j)$ $\forall k, j$

For eg. $a = 9$ AM, $b = 10$ AM, $c = 11$ AM, $d = 12$ AM

in 9-10 AM doesn't affect # in 11-12 AM.

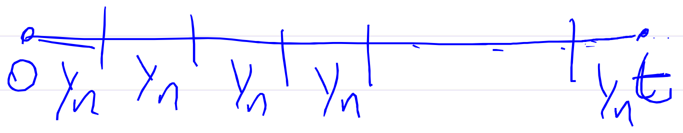
Fix t. $N(t)$ = # customers arriving by time t

= # in $[0, t]$.

So $N(t)$ is a R.V. taking values in $\mathbb{N}^*$

What is a good pmf for $N(t)$? $P_X(k) = ?$ $X = N(t)$

Let $n \gg 1$ be large.



; $t \in \mathbb{N}$.

Y_i = # of customers arriving in interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ $\quad i \leq tn$

# of intervals $= tn$. since $t \in \mathbb{N}$.

All intervals are eq. length.

By (2), $Y_i \in \{0,1\}$ two customers can't arrive in $\left[\frac{i-1}{n}, \frac{i}{n}\right]$.

By (1), $E[Y_i] = \frac{\lambda}{n}$. $\Rightarrow Y_i \overset{d}{=} Ber\left(\frac{\lambda}{n}\right)$. for any i

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As if each interval, we are tossing a coin with prob. $\frac{\lambda}{n}$.

(3) $\Rightarrow$ coin tosses are indep.

$N(t) \approx$ # of heads in tn many coin-tosses with prob. λ/n

$= Bin\left(nt, \frac{\lambda}{n}\right) \overset{d}{=} X_n$ [Think what happens if $t \notin \mathbb{N}$]

Fix k.

$\mathbb{P}(N(t) = k) \approx \mathbb{P}(X_n = k)$

$$= \binom{nt}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{nt-k} = a_n$$

$$\lim_{n\to\infty} a_n = \frac{(nt)!}{(nt-k)!}\,\frac{1}{n^k}\left[\left(1-\frac{\lambda}{n}\right)^n\right]^t \frac{\lambda^k}{k!}\,\frac{1}{\left(1-\frac{\lambda}{n}\right)^k}$$

$\left(1-\frac{\lambda}{n}\right)^n \to e^{-\lambda}$; $\quad \frac{1}{\left(1-\frac{\lambda}{n}\right)^k} \to 1$ as $n \to \infty$.

$$\approx \frac{\lambda^k}{k!} e^{-\lambda t}\,\frac{(nt)!}{(nt-k)!}\,\frac{1}{n^k}$$

$$\approx \frac{\lambda^k}{k!} e^{-\lambda t}\,\frac{nt\,(nt-1)\cdots(nt-k+1)}{n\cdot n \cdots n}$$

$$\approx \frac{(\lambda t)^k}{k!} e^{-\lambda t}\; 1\left(1-\frac{1}{nt}\right)\left(1-\frac{2}{nt}\right)\cdots\left(1-\frac{k-1}{nt}\right)$$

$$\longrightarrow \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

$$P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k \geq 0.$$

$p_X(k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$, Check this is a pmf

& X is called Poisson (λt) r.v.. Ex. $E[X]$, $VAR[X]$.