14/12 LECTURE 24 : CDF & PROPERTIES.

Eg 24.1 N = # of customers arriving by time 1 ie, # in $[0,1]$. [Poisson RV]

Reasonable to assume $N \overset{d}{=} \text{Poi}(\lambda)$ (Poisson) i.e., $p_N(k) = \frac{\lambda^k e^{-\lambda}}{k!}$, $k \geq 0$, $\lambda > 0$.

Check $\sum_{k=0}^{\infty} p_N(k) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{\lambda} e^{-\lambda} = 1$.

$$\mathbb{E}[N] = \sum_{n=0}^{\infty} n\, p_N(n) = \sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda} = \lambda \sum_{n=1}^{\infty} \frac{n \lambda^{n-1}}{n!} e^{-\lambda}$$

$$= \lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} e^{-\lambda} = \lambda.$$

Compute $\text{VAR}[N]$; $\mathbb{E}[N^k]$, $k \geq 1$.

Eg 24.2 . N is same as above. Customers are happy w.p. p & not happy w.p. $(1-p)$ & $\Big\}$ indep of others.

H = # of happy customers. $p_H(\cdot) = ?$

$\mathbb{E}(H)$ "=" λp (guess)

$$p_H(k) = \mathbb{P}(H = k)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(\{H=k\} \cap \{N=n\}) \quad \text{(law of total prob.)}$$

$$\left(\Omega = \bigsqcup_{n=0}^{\infty} \{N=n\}\right)$$

$$\mathbb{P}(\{H=k\} \cap \{N=n\}) = \mathbb{P}(H=k \mid N=n)\, \mathbb{P}(N=n) \quad \text{(conditional prob formula)}$$

$\underbrace{\{H=k\}}_{\#\text{h. cust} = k}$ $\underbrace{\{N=n\}}_{\#\text{cust} = n}$

$$= \mathbb{P}(\#\text{h. cust} = k \mid \text{Total \# of cust} = n)\, p_N(n).$$

$$= \mathbb{P}(\text{Bin}(n,p) = k)\, p_N(n)$$

$$= \binom{n}{k} p^k (1-p)^{n-k} e^{-\lambda} \frac{\lambda^n}{n!} \quad \text{(pmfs of Bin \& Pois)}$$

$$= \frac{e^{-\lambda p} (\lambda p)^k}{k!} \frac{(\lambda(1-p))^{n-k}}{(n-k)!} e^{-\lambda(1-p)} \quad \text{— (1)}$$

Note that $\mathbb{P}(\{H=k\} \cap \{N=n\}) = 0$ if $n < k$.

So $\quad p_H(k) = \sum_{n=k}^{\infty} \mathbb{P}(\{H=k\} \cap \{N=n\})$

$$= \frac{e^{-\lambda p}(\lambda p)^k}{k!} \sum_{n=k}^{\infty} \frac{(\lambda(1-p))^{n-k}}{(n-k)!} e^{-\lambda(1-p)}$$

$$= \frac{e^{-\lambda p}(\lambda p)^k}{k!} \qquad H \overset{d}{=} \mathrm{Poi}(\lambda p)$$

$$\Downarrow$$

$$\mathbb{E}[H] = \lambda p.$$

$H' = N - H = \#$ not happy customers.

$$H' \overset{d}{=} \mathrm{Poi}(\lambda(1-p))$$

Fix k, j

$$\mathbb{P}(H=k, H'=j) = \mathbb{P}(\{H=k\} \cap \{H'=j\})$$

$$= \mathbb{P}(\{H=k\} \cap \{N=k+j\}) \quad (N = H + H')$$

$$= \frac{(\lambda p)^k}{k!} e^{-\lambda p} \frac{(\lambda(1-p))^j}{j!} e^{-\lambda(1-p)} \quad (\text{from } 1)$$

$$= \mathbb{P}(H=k)\,\mathbb{P}(H'=j). \quad [\text{Thinning prop.}]$$

If $N \overset{d}{=} \mathrm{Poi}(\lambda)$, then $\forall k, j$ & $p \in [0,1]$

$$\mathbb{P}(H=k, H'=j) = \mathbb{P}(H=k)\,\mathbb{P}(H'=j). \quad \text{--- } 2$$

Ex** Suppose N is a r.v. with values in $\mathbb{N}^*$ & $\forall k, j$ & $p \in [0,1]$, N satisfies 2. Is N a Poisson r.v.?

Defn: [Cumulative Distribution Function (CDF)]

Let X be a r.v. with pmf p_X. CDF of X is a function $F_X : \mathbb{R} \to [0,1]$ & defined as

$$F_X(x) := \mathbb{P}(X \le x) = \sum_{y : y \le x} p_X(y)$$

[Note: Given a pmf $p_X : \mathbb{R} \to [0,1]$, CDF exists].

($\sum_x p_X(x) = 1 \Rightarrow p_X \neq 0$ on atmost countably many values)

For eg. $p_X(x) = \frac{1}{x}, x > 0$. $p_X(x) \longrightarrow 0$ as $x \to \infty$.

But $\sum_x p_X(x)$ isn't defined !

If we write $\sum_x p_X(x)$, it implies that $p_X \neq 0$ on atmost a countable sets

Eg 24.4 (1) $X \stackrel{d}{=} \text{Unif}[n]$ $\quad p_X(k) = \frac{1}{n}, k \in \{1, \cdots, n\}$

$$F_X(x) = \begin{cases} 0 & -\infty < x < 1 \\ 1/n & 1 \le x < 2 \\ 2/n & 2 \le x < 3 \\ \vdots & \\ n-1/n & n-1 \le x < n \\ 1 & n \le x < \infty. \end{cases}$$

(2) $X \stackrel{d}{=} \text{Ber}(p)$ $\quad p_X(0) = 1-p = 1 - p_X(1)$.

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$\boxed{\text{THM } 24.5}$ (Prop. of CDF). Let X be a r.v. with CDF F_X.

(1) $F_X(x) \le F_X(y) \quad \forall \; x \le y$ (Monotonicity)

(2) $\lim_{x \to \infty} F_X(x) = 1$; $\lim_{x \to -\infty} F_X(x) = 0$.

(3) $F_X(x+) := \lim_{y \downarrow x} F_X(y) = F_X(x)$. $\forall \; x \in \mathbb{R}$ (F_X is R.C.)

(4) $p_X(x) = \mathbb{P}(X = x) = F_X(x) - F_X(x-)$

$= F_X(x) - \lim_{y \uparrow x} F_X(y) \quad \forall \; x \in \mathbb{R}$

(5) $\mathbb{P}(x < X \le y) = F_X(y) - F_X(x)$, $x < y$.

Proof (1) $F_X(x) = \sum_{z : z \le x} p_X(z) \le \sum_{z: z \le y} p_X(z) = F_X(y)$.

(defn of CDF) ($p_X \geq 0$) defn of CDF

(2) $F_X(x) = P(X \le x)$.

Take $x_n \to \infty$ as $n \to \infty$. ETPT $\lim\limits_{n\to\infty} F_X(x_n) = 1$.

$(\Rightarrow \lim\limits_{x \to \infty} F_X(x) = 1)$

Given $\{x_n\}$, pick subsequence $y_n \ni y_n \uparrow \infty$.

$F_X(y_n) = \mathbb{P}(X \le y_n) = P[X \in (-\infty, y_n])$,

$\le \mathbb{P}(X \in (-\infty, y_{n+1}])$

Let $A_n = \{X \in (-\infty, y_n]\}$. Then since $y_n \uparrow \infty$

$A_n \uparrow A = \bigcup\limits_{n\ge1} A_n = \bigcup\limits_{n \ge 1} \{X \in (-\infty, y_n]\}$

$= \{X \in \bigcup\limits_{n\ge1} (-\infty, y_n]\}$

[Proven before in class] $= \{X \in \mathbb{R}\}$ as $y_n \uparrow \infty$

By c. add. $\mathbb{P}(A_n) \uparrow \mathbb{P}(A) = \mathbb{P}(X \in \mathbb{R}) = 1$.

Let $Y = -X$. $F_X(x) = P(\cancel{X} \le x) = P(-Y \le x)$

$= \mathbb{P}(Y \geq -x) \le \mathbb{P}(Y > -x)$ monotonicity

$= 1 - \mathbb{P}(Y \le -x)$

$= 1 - F_Y(-x)$

So $F_X(x) \le 1 - F_Y(-x)$ (since limit exists.)

$0 \le \overline{\lim\limits_{x \to -\infty}} F_X(x) \le 1 - \underline{\lim\limits_{x\to-\infty}} F_Y(-x) = 1 - \lim\limits_{y\to\infty} F_Y(y) = 0.$

Since $\overline{\lim_{x\to-\infty}} F_X(x) = 0$ & $F_X \geq 0$, by prev. part.

$$\lim_{x\to-\infty} F_X(x) = 0.$$

(3) T.S.T. $\lim_{y\downarrow x} F_X(y) = F_X(x)$.

Let $y_n \downarrow x$ as $n\to\infty$. $A_n = \{X \leq y_n\} \downarrow A = \bigcap_{n\geq 1} \{X \leq y_n\}$

Again we have shown that $\mathbb{P}(A_n) \downarrow \mathbb{P}(A)$ as $n\to\infty$.

$= \{X \leq y_n \ \forall n \geq 1\}$

$= \{X \leq x\}$

$\Rightarrow \underbrace{F_X(y_n) = \mathbb{P}(A_n)}_{\text{defn}} \underbrace{\downarrow}_{\text{proven above}} \underbrace{\mathbb{P}(A) = F_X(x)}_{\text{defn}}.$

(4) T.S.T. $p_X(x) = F_X(x) - F_X(x-)$

Let $y_n \uparrow x$ as $n\to\infty$. $(y_n < x, \forall n \geq 1)$ $A_n = \{X \leq y_n\} \uparrow A = \bigcup_{n\geq 1} \{X \leq y_n\}$

$= \{X \leq y_n \text{ for some } n \geq 1\} = \{X < x\}$

Again $F_X(y_n) = \mathbb{P}(A_n) \uparrow \mathbb{P}(A) = \mathbb{P}(X < x)$.

$\Rightarrow \lim_{y\uparrow x} F_X(y) = F_X(x-) = \mathbb{P}(X < x)$.

So $F_X(x) - F_X(x-) = \mathbb{P}(X \leq x) - \mathbb{P}(X < x)$

$= \mathbb{P}(\{X \leq x\} \setminus \{X < x\})$

$= \mathbb{P}(X = x) = p_X(x)$.

[observe F_X is cts at x iff $p_X(x) = 0$]

(5) $\mathbb{P}(x < X \leq y) = \mathbb{P}(X \in (x, y])$, $x < y$.

$$= \mathbb{P}(\{X \in (-\infty, y]\} \setminus \{X \in (-\infty, x]\})$$

$$= \mathbb{P}(X \in (-\infty, y]) - \mathbb{P}(X \in (-\infty, x])$$

$$((-\infty, x] \subseteq (-\infty, y])$$

(defn of CDF) $= F_X(y) - F_X(x)$.

∎

Check:

$$\mathbb{P}(x \le X \le y) = F_X(y) - F_X(x) + p_X(x) = F_X(y) - F_X(x-)$$

$$\mathbb{P}(x \le X < y) = F_X(y) - F_X(x) + p_X(x) - p_X(y)$$

$$\mathbb{P}(x < X < y) = F_X(y) - F_X(x) - p_X(y) = F_X(y-) - F_X(x)$$

COR. 21.6 Let X be a discrete r.v. with CDF F_X. Then the pmf p_X is given by $p_X(x) = F_X(x) - F_X(x-)$ $\forall x \in \mathbb{R}$.

Defn 21.7: A fn. $F : \mathbb{R} \to \mathbb{R}$ satisfying the following 3 conditions is called a DF (distribn. fn.)

(1) $F(x) \le F(y)$ $\forall x \le y$ (Monotonicity)

(2) $\lim_{x \to \infty} F(x) = 1$; $\lim_{x \to -\infty} F(x) = 0$ (∞ & $-\infty$ limits)

(3) $F(x+) = \lim_{y \downarrow x} F(y) = F(x)$. (Right cty)

- CDF F_X of a discrete r.v. X is a DF!

Eg: If X is a discrete r.v. with values in $\mathbb{N}^*$ then

$$F_X(x-) = \sum_{y : y < x} p_X(y) = \begin{cases} F_X(x-1) & \text{if } x \in \mathbb{N}^* \\ F_X(\lfloor x \rfloor) & x \notin \mathbb{N}^* \end{cases}$$

$\hookrightarrow$ floor of x.