17/12 LECTURE 25 :

X – r.v. – $X \in \mathbb{R}$. – takes values in $\mathbb{R}$.

Discrete r.v.	Cts r.v.
(1) X takes countably many values.	(1) X takes uncountably many values
(2) Formal description – pmf $p_X(x) = \mathbb{P}(X = x)$	(2) pdf – f_X – piecewise cts $\mathbb{P}(X \in (a,b)) = \int_a^b f_X(x)\,dx$
(3) CDF: $F_X(x) = \mathbb{P}(X \le x)$ $p_X(x) = F_X(x) - F_X(x-)$	(3) $F_X(x) = \mathbb{P}(X \le x)$ $F_X'(x) = f_X(x)$ if f_X is cts at x.
CDF gives pmf everywhere.	CDF gives pdf except at finitely many points.

X cts r.v. $\iff$ $\exists$ a pdf f_X ($f_X : \mathbb{R} \to [0, \infty)$ p.cts, $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$)

$\iff$ $\exists$ a CDF $F_X : \mathbb{R} \to [0,1]$ $\ni$ F_X' exists except at fin. many pts & F_X' is a pdf.

- Two pdfs f & g are equal ($f \equiv g$) if $f = g$ except at finitely many points.
- Two cts r.v. X & Y are equal in distribution ($X \stackrel{d}{=} Y$) if $f_X \equiv f_Y$.
- $f_X \equiv f_Y \implies F_X(x) = F_Y(x) \ \forall x \in \mathbb{R}$.

Eg 25.1

X – Choose a point at random in $[0,1]$.

So $\mathbb{P}(X \in (a,b)) = b - a \qquad 0 \le a < b \le 1$.

" prob. random point is in (a,b)

So $f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else.} \end{cases}$ pdf

X as above is called UNIFORM $([0,1])$ r.v. ($X \stackrel{d}{=} \text{Unif}[0,1]$)

CDF $F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1. \end{cases}$

Eg 25.2 X – Choose a random point uniformly in (a,b), $a<b$, $a,b \in \mathbb{R}$.

So $\mathbb{P}(X \leq c) = \begin{cases} 0 & c \leq a \\ 1 & c \geq b \end{cases}$

$\parallel$

Prob. random pt $\leq c$

$a<c<b$, $\mathbb{P}(X \leq c) \underset{\text{proportional to}}{\propto} c-a$, $\mathbb{P}(X \leq a) = 0 = 1 - \mathbb{P}(X \leq b)$

So $\mathbb{P}(X \leq c) = \dfrac{c-a}{b-a}$

Check. $F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \leq x < b \\ 1 & x \geq b \end{cases}$ is a DF.

F_X is CDF of X. X is called UNIFORM(a,b) r.v.

$X \overset{d}{=} \text{UNIF}(a,b) \overset{d}{=} U(a,b)$.

pdf $f_X(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{elsewhere.} \end{cases}$

Eg 25.3 Q of customers arriving. T – time of arrival of the first customer.

$N(t) = \#$ of customers upto time $t \overset{d}{=} \text{Poi}(\lambda t)$

$\mathbb{P}(N(t) = k) = e^{-\lambda t} \dfrac{(\lambda t)^k}{k!}$, $k = 0, 1, \ldots$

$\{T > t\}$ = No customer upto time t. $t \geq 0$.

$= \{N(t) = 0\}$

$\mathbb{P}(T \leq t) = 1 - \mathbb{P}(T > t) = 1 - \mathbb{P}(N(t) = 0)$

$= 1 - e^{-\lambda t}$, $t \geq 0$

So CDF of T is $F_T(t) = 1 - e^{-\lambda t}$, $t \geq 0$

pdf of T is $f_T(t) = \lambda e^{-\lambda t}$, $t \geq 0$.

T is called EXPONENTIAL (λ) r.v. ($T \stackrel{d}{=} EXP(\lambda)$).

T_k – Time of arrival of k^{th} customers. $F_{T_k}(\cdot)$ & $f_{T_k}(\cdot)$

Eg 25.4

X – Standard Normal r.v. / Gaussian r.v. ($X \stackrel{d}{=} N(0,1)$)

if $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in \mathbb{R}$.

To check this is a pdf.

$$I = \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = 1 \ ?$$

(Set $x = r\cos\theta$ $y = r\sin\theta$)

$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x^2+y^2)}{2}} dx\, dy \overset{\text{Polar co-ordinate}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\infty} r e^{-r^2/2} dr\, d\theta$$

$$= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\right) \times \left(\int_0^{\infty} r e^{-r^2/2} dr\right) = 1.$$

RE-VISIT RULES:

(1) $\mathbb{P}(X \in (a,b]) \overset{\text{by defn of CDF}}{=} F_X(b) - F_X(a) \overset{\text{FTC}}{=} \int_a^b f_X(x)\, dx$

(2) $\mathbb{P}(X \in \bigcup_{i=1}^{\infty} (a_i, b_i]) = \sum_{i=1}^{\infty} (F_X(b_i) - F_X(a_i))$ (Countable additivity ASSUME)

$= \sum_{i=1}^{\infty} \int_{a_i}^{b_i} f_X(x)\, dx$

(3) $\mathbb{P}(X \in \bigcup_{i=1}^{\infty} (a_i, b_i]) \leq \sum_{i=1}^{\infty} \mathbb{P}(X \in (a_i, b_i])$ (Count. subadd)

$= \sum_{i=1}^{\infty} F_X(b_i) - F_X(a_i)$.

(4) $\mathbb{P}(X \in A) \le \mathbb{P}(X \in B)$ when A, B are intervals & $A \subseteq B$.

(5) $\mathbb{P}(X \in A \cup B) = \mathbb{P}(X \in A) + \mathbb{P}(X \in B) - \mathbb{P}(X \in A \cap B)$ A, B are intervals.

(6) I-E & Bonferroni all hold for intervals!!!

$\mathcal{I}$ = Intervals = $\{ (a,b], [a,b), (a,b), [a,b] : -\infty \le a < b \le \infty \}$

Eg: 25.5 Let $X \overset{d}{=} U(0,1)$. Define $Y = aX + b$, $a > 0$, $b \in \mathbb{R}$.
What is $Y = ?$

$$F_Y(x) := \mathbb{P}(Y \le x) = \mathbb{P}(aX + b \le x)$$
$$= \mathbb{P}(aX \le x - b) = \mathbb{P}\left(X \le \frac{x-b}{a}\right)$$

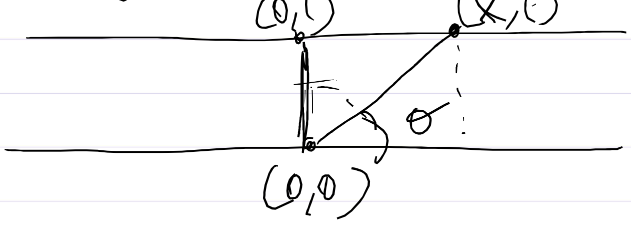
$$= \begin{cases} 0 & x - b < 0 \iff x < b \\ \frac{x-b}{a} & 0 \le x - b < a \iff b \le x < b+a \\ 1 & x - b \ge a \iff x \ge b + a. \end{cases}$$

$$f_Y(x) = F_Y'(x) = \begin{cases} 1/a & b \le x \le b+a \\ 0 & \text{else.} \end{cases}$$

$\Rightarrow Y \overset{d}{=} \mathrm{Unif}(b, b+a)$

Ex. Let $X \overset{d}{=} N(0,1)$. What is $Y = aX + b$? $a > 0$, $b \in \mathbb{R}$.

Eg: 25.6 Let θ be an angle chosen uniformly at random in $(0, \pi)$.



X = r.v.

$\theta \stackrel{d}{=} \text{Unif}(0,\pi)$ $\qquad X = \tan(\frac{\pi}{2} - \theta)$

$\theta \in (0,\pi) \Rightarrow \frac{\pi}{2} - \theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$.

$\Rightarrow \tan(\frac{\pi}{2} - \theta)$ is strictly decreasing. on $(0,\pi)$

$\Rightarrow$ inverse $\arctan(x) \in (\frac{-\pi}{2}, \pi/2)$

$$
\begin{aligned}
F_X(x) &= \mathbb{P}(X \le x) = \mathbb{P}(\tan(\tfrac{\pi}{2} - \theta) \le x) \\
&= \mathbb{P}(\tfrac{\pi}{2} - \theta \le \arctan(x)) \quad \text{as } \tan(\tfrac{\pi}{2} - \theta) \text{ is strictly } \downarrow \\
&= \mathbb{P}(\theta \ge \tfrac{\pi}{2} - \arctan(x)) \\
&= 1 - \mathbb{P}(\theta < \tfrac{\pi}{2} - \arctan(x)) \\
&= 1 - F_\theta(\tfrac{\pi}{2} - \arctan(x)) \quad \text{since } \theta \text{ is cts } F_\theta \text{ is cts} \\
&= 1 - \tfrac{1}{\pi}(\tfrac{\pi}{2} - \arctan(x)) \\
&= \tfrac{1}{2} + \tfrac{1}{\pi}\arctan(x), \quad x \in \mathbb{R}
\end{aligned}
$$

What is the pdf ? $\quad f_X(x) = F_X'(x) = \dfrac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}$.

X is called Cauchy r.v.

$Y = aX + b$?

PREVIEW: (1) Given $h: \mathbb{R} \to \mathbb{R}$ diff'ble & nice.

what is pdf/CDF of $h(X)$?

(2) $h: \mathbb{R} \to \mathbb{R}$ cts; $\quad \mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x)\,dx$.

if well-defined.