16/12 LECTURE 25 – CTS RV – INTRO.

Discrete prob. space Ω – Countable set. Define $p(\omega) \geq 0, \omega \in \Omega$

$\ni \sum_{\omega} p(\omega) = 1.$

If Ω is uncountable what is $\sum_{\omega} p(\omega)$?

Eg 25.1 $\Omega = \{0,1\}^{\mathbb{N}}$ – ∞'ly many coin tosses. $\omega \in \Omega$ $\omega = (\omega_1, \omega_2, \dots)$

WANT TO DEFINE $\mathbb{P}$ – corresponding to ∞'ly many fair coin tosses.

$A_n = \{\omega \in \{0,1\}^{\mathbb{N}} : \omega_1 = \dots = \omega_n = 1\}$

$P(A_n)$ "=" 2^{-n}

But is there a $p \ni \mathbb{P}(A_n) = \sum_{\omega \in A_n} p(\omega)$?

If $\exists p$, $p(\omega)$ "=" $P(\{\omega\})$ $\forall \omega \in \Omega$.

$= 0$ (intuitive guess).

[For eg. take $\omega = (1, \dots, 1, \dots)$ $\mathbb{P}(\{\omega\}) \leq \mathbb{P}(A_n)$ $\forall n$]

But $\mathbb{P}(A_n) \neq 0$?

Eg 25.2 $\Omega = [0,1]$. Pick a random point from Ω.

Let X be the random point

$\mathbb{P}(X \in [a,b])$ "=" $b-a$ $\quad 0 \leq a < b \leq 1.$

so $\mathbb{P}(X \in [a,b]) = \int_a^b \mathbb{P}(X = x)\, dx$?

If we take $\mathbb{P}(X = x) = 1$ $\forall x$, then ok ?? [Not POSSIBLE

Intuition $\Rightarrow$ $\mathbb{P}(X = x) \leq \mathbb{P}(X \in (x-\varepsilon, x+\varepsilon))$ "=" 2ε, $\forall \varepsilon > 0$

$\Rightarrow \mathbb{P}(X = x) = 0.$

So $\mathbb{P}(X \in [a,b]) = \int_a^b f(x)\,dx$ but $\mathbb{P}(X = x) \neq f(x)$?

→ We'll not define prob. on uncountable spaces but just focus on RV's on $\mathbb{R}$ taking uncountably many values.

ET to define RV's & not Prob. spaces. (So not a restriction)

RESTRICTION — $\Omega = \mathbb{R}$!

Why? We know properties of functions on $\mathbb{R}$ – Cty, diff & integrations

Defn 25.3 A DF is a fn $F : \mathbb{R} \longrightarrow [0,1]$ s.t.

(i) $F(s) \le F(t) \quad \forall\ s \le t$ (increasing)

(ii) $\lim_{s \to -\infty} F(s) = 0 = 1 - \lim_{s \to \infty} F(s)$ (0 – 1 limits)

(iii) $F(t+) = F(t)$ (R.C.)

Cor 25.4. If X is a discrete r.v. taking values in $\mathbb{R}$ (i.e. $p_X(x) \ge 0$, $\sum_x p_X(x) = 1$) then the CDF of X, F_X is a DF.

(Done in last class).

$\left(F_X(x) := \sum_{y \le x} p_X(y)\right)$.

— $p_X(x) = F_X(x) - F_X(x-)$.

Cor 25.5 Let F be a DF & assume that $\{x : F(x) - F(x-) \ne 0\}$ is countable. Let $< x_{-2} < x_{-1} < 0 \le x_1 < x_2 < \cdots$ ($0 = x_0$).

We also assume that $F(x) = F(y)$ if $x, y \in (x_k, x_{k+1})\ \forall k \in \mathbb{Z}$.

(i.e., F is a step function with jumps at x_k, $k \in \mathbb{Z}$)

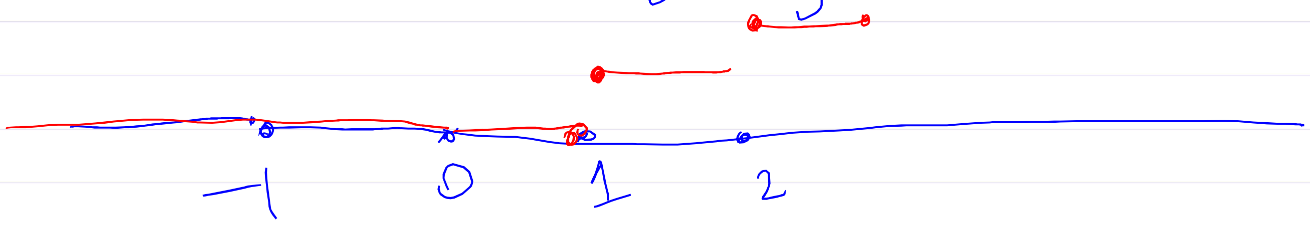
Then $\exists$ a discrete r.v. $X \ni F_X = F$.

For simplicity assume there are no jumps in $(-\infty, 0)$

Proof in pictures: Let $x_1, \ldots, x_n, \ldots$ be $F(x_i) - F(x_i-) \ne 0$.

see next page

Eg: $F(x) = \max\{\lfloor x \rfloor, 0\}$ — floor of x. is a step fn with countably many jumps.



Contd. of $p_X(x_i) := F(x_i) - F(x_i -)$.

$F \uparrow \Rightarrow p_X(x_i) \geq 0$.

We need to check $\sum_{x_i} p_X(x_i) = 1$. [Ex.]

S.T. $\sum_{i=1}^{\infty} p_X(x_i) = 1 - F(0)$; $\sum_{i=1}^{\infty} p_X(x_{-i}) = F(0)$

If DF is a step function with countably many jumps then it is CDF of a discrete r.v. !

We can use DF's to define discrete r.v. !

as any DF, CDF of a discrete r.v. ?

Eg. 25.6

$$F(s) = \begin{cases} 0 & s < 0 \\ s & 0 \leq s \leq 1 \\ 1 & s \geq 1 \end{cases}$$

Check F is a DF. But $\nexists$ no discrete r.v. $X \ni F_X = F$!

Defn 25.7 Let $f : \mathbb{R} \to [0, \infty)$ be a piece-wise continuous function (i.e., discontinuous at finitely many points) $\ni \int_{-\infty}^{\infty} f(x)\,dx = 1$. Such a f is called a prob. density function (pdf).

Say f is discts at $a_1 < a_2 < \cdots < a_k$.

then $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a_1} f(x)\,dx + \int_{a_1}^{a_2} f(x)\,dx + \cdots + \int_{a_k}^{\infty} f(x)\,dx$.

on each (a_i, a_{i+1}), f is cts & so $\int_{a_i}^{a_{i+1}} f(x)\,dx$ is well-defined!

Eg: 25.8 (1) $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$; f is a pdf

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{0} f(x)\,dx}_{=0} + \underbrace{\int_{0}^{1} f(x)\,dx}_{=1} + \underbrace{\int_{1}^{\infty} f(x)\,dx}_{=0}.$$

$\Rightarrow f$ is a pdf.

(2) $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0 & x < 0 \end{cases}$, $\lambda > 0$.

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{0} f(x)\,dx}_{=0} + \lambda\int_{0}^{\infty} e^{-\lambda x}\,dx$$

$$= 0 + \lambda \cdot \frac{1}{\lambda} = 1.$$

$\Rightarrow f$ is a pdf.

LEMMA 25.9 If f is a pdf, then $F(x) := \int_{-\infty}^{x} f(t)\,dt$, $x \in \mathbb{R}$ is a DF.

Proof: (i) Let $x \le y$. $F(x) = \int_{-\infty}^{x} f(t)\,dt \le \int_{-\infty}^{y} f(t)\,dt = F(y)$

$$\underset{\text{defn}}{=} \qquad \underset{f \ge 0}{\le} \qquad \underset{\text{defn}}{=}$$

(ii) $\lim_{x\to\infty} F(x) \underset{\text{defn}}{=} \lim_{x\to\infty} \int_{-\infty}^{x} f(t)\,dt \underset{f\ge 0 \text{ \& defn of } \int}{=} \int_{-\infty}^{\infty} f(t)\,dt \underset{f \text{ is pdf}}{=} 1.$

(ii)ly $\lim_{x \to -\infty} F(x) = 0.$

(iii) $F(x+) = \lim_{h \downarrow 0} F(x+h) = \underset{h \downarrow 0}{\text{Lt}} \int_{-\infty}^{x+h} f(t)\,dt$

$\left[F(x+) := \underset{y \downarrow x}{\text{Lt}} F(y) = \underset{y \to x+}{\text{Lt}} F(y) = \underset{h \to 0+}{\text{Lt}} F(x+h) \right]$

$$= \underset{h \downarrow 0}{\text{Lt}} \left(\int_{-\infty}^{x} f(t)\,dt + \int_{x}^{x+h} f(t)\,dt \right)$$

$$= \int_{-\infty}^{x} f(t)\,dt + \underset{h \downarrow 0}{\text{Lt}} \int_{x}^{x+h} f(t)\,dt$$

$$= \int_{-\infty}^{x} f(t)\,dt + 0$$

Check $F(x-) = F(x)$. i.e., left-cts & so cts.

By Fund. thm of calculus, $F'(x) = f(x)$ except when x is a pt of discty of f.

THM 25.10 Let f be a pdf & F the corresponding DF. Then $\exists$ a r.v. $X \ni F(t) = \mathbb{P}(X \le t) \ \forall t \in \mathbb{R}$.

[WE WILL NOT SEE PROOF !!!]

X as above is called a $\boxed{\text{CTS RV.}}$ & F is its CDF & f is its pdf.

Discrete r.v.'s $X \iff$ pmf $p_X \iff$ step fn DF's. F_X

Cts r.v.'s $X \iff$ pdf $f_X \iff F_X' \text{``}=\text{''} f_X$ DF's F_X

except at finitely many pts.

RULES & TOOLS TO WORK WITH CTS RV'S.

(1) $\mathbb{P}(X \in (a,b]) = F(b) - F(a)$

$\mathbb{P}(X \in (a,b)) = \mathbb{P}(X \in [a,b]) = \mathbb{P}(X \in [a,b))$

(2) if $I = \bigcup_{j=1}^{\infty} (a_j, b_j]$ (I is a union of countably many disjoint intervals)

then $\mathbb{P}(X \in I) = \sum_{j=1}^{\infty} F(b_j) - F(a_j)$.

Rule (2) not APPLICABLE TO GENERAL SETS !!!

$\mathbb{P}(X \in A)$ NOT DEFINED $\forall A$!!!