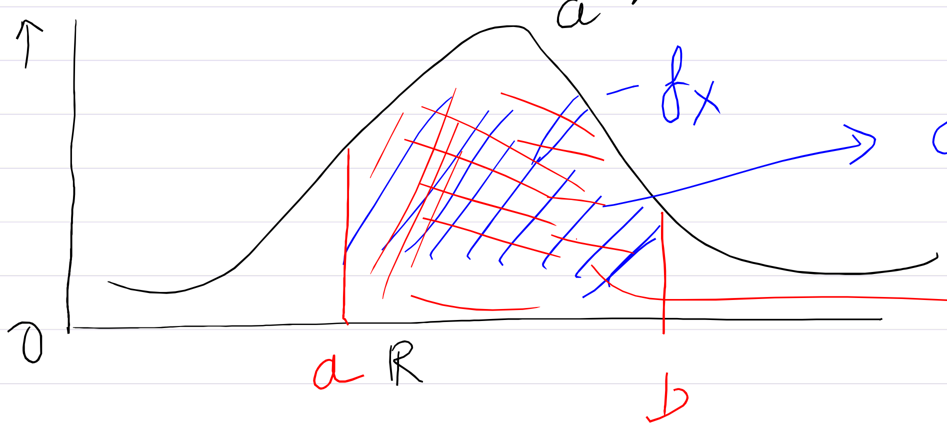
21/12 LECTURE 26 : TRANSFORMATIONS OF R.V.

X cts r.v. $\iff \exists$ a pdf f_X ($f_X : \mathbb{R} \to [0,\infty)$ p.cts, $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$)

$\iff \exists$ a cts CDF $F_X : \mathbb{R} \to [0,1] \ni F_X'$ exists except at fin. many pts & F_X' is a pdf.

$$\mathbb{P}(X \in (a,b]) = \int_a^b f_X(x)\,dx = F_X(b) - F_X(a) \qquad -\infty \le a < b < \infty$$

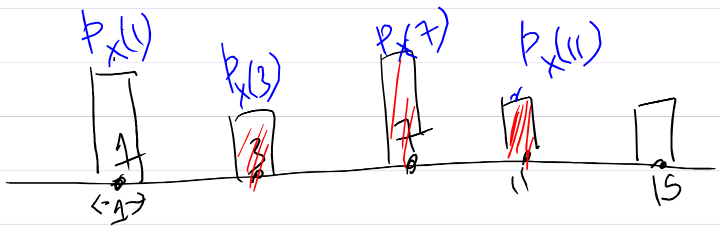
Say f_X is cts.



area under the curve = 1

$\mathbb{P}(X \in (a,b])$ " = " area under curve btw a & b.

Discrete X – pmf P_X



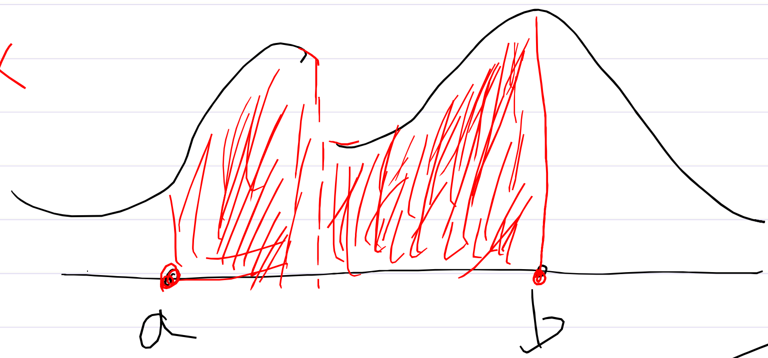
width = 1.

area of bar at i = $p_X(i)$

$$\mathbb{P}(X \in (1, 11]) = \sum_{1 < i \le 11} p_X(i)$$

" area of shaded region. = Total area of bars btw 1 & 11 (incl.)

X cts r.v. f_X p-cts

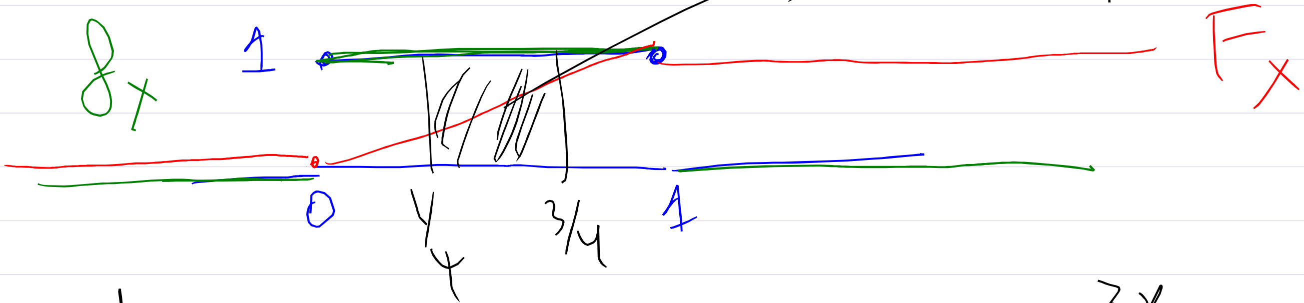


$\mathbb{P}(X \in (a,b])$ = area of shaded region.

Plot CDF's

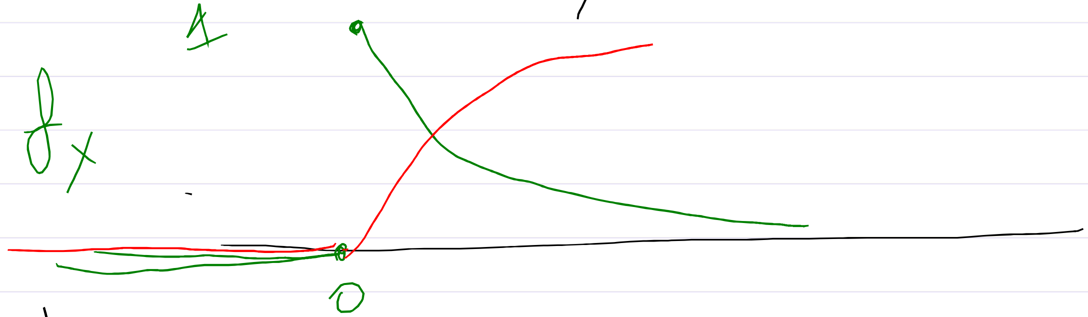
F_X

Eg 26.1 Uniform $-$ $X \sim U(0,1)$ $\mathbb{P}(X \in (\frac{1}{4}, \frac{3}{4}])$

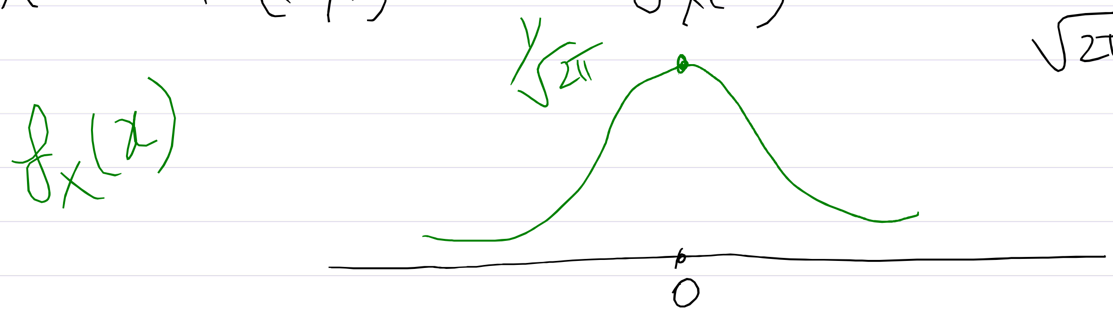


Eg 26.2 $X \stackrel{d}{=} \text{EXP}(\lambda)$ $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$

$F_X(x) = 1 - e^{-\lambda x}$, $x \geq 0$



Eg 26.3 $X \stackrel{d}{=} N(0,1)$ $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in \mathbb{R}$



Bell curve
Gaussian curve

Qn: $h : \mathbb{R} \to \mathbb{R}$ cts fn. X - cts r.v.

Let $Y = h(X)$. If Y cts? What is the pdf?

LEMMA 26.4 Let $Y = aX + b$, $a > 0$ & $b \in \mathbb{R}$. X is cts r.v.

Then Y is cts r.v. & $f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$.

Proof: $F_Y(y) = \mathbb{P}(Y \leq y)$

$= \mathbb{P}(aX + b \leq y) = \mathbb{P}\left(X \leq \frac{y-b}{a}\right)$

$= F_X\left(\frac{y-b}{a}\right)$ (Check F_Y is a CDF).

Since F_X is cts, so is F_Y.

If $F_X'\left(\frac{y-b}{a}\right)$ exists, then by chain rule

$F_Y'(y) = \frac{1}{a} F_X'\left(\frac{y-b}{a}\right) \underset{(\text{FTC})}{=} \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$

Since F_X isn't diff'ble only at fin. many pts ($\because f_X$ is p.cts)
So is F_Y. Hence $f_Y(y) = F_Y'(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$
whenever $\frac{y-b}{a}$ isn't a pt. of discty of f_X.
[$F_X'(x)$ exists iff x is a cty point of f_X]
Clearly f_Y is p.cts. (f_Y is cts at $y \iff f_X$ is cts at $\frac{y-b}{a}$)

$$\frac{1}{a}\int_{-\infty}^{\infty} f_X\left(\frac{y-b}{a}\right) dy \underset{\text{CoV}}{=} \frac{1}{a}\int_{-\infty}^{\infty} a\, f_X(x)\, dx \qquad \text{set } x = \frac{y-b}{a}, \quad dy = a\,dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

So $F_Y' = f_Y$ is a pdf of Y. ∎

Alternate – Guess f_Y & Check $\mathbb{P}(Y \le y) = \int_{-\infty}^{y} f_Y(y)\,dy$!

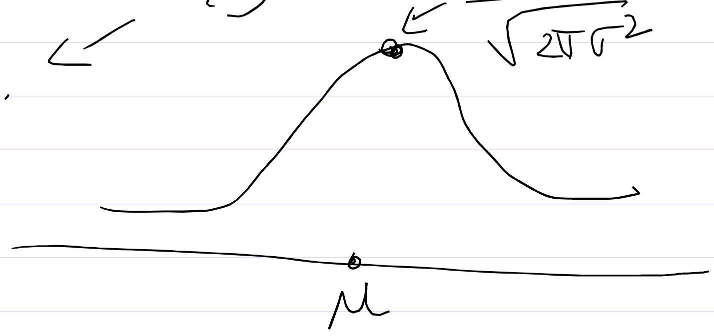
Eg 2605 (1) $X \overset{d}{=} N(0,1)$ $\quad f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$Y = \sigma X + \mu$, $\mu \in \mathbb{R}$, $\sigma > 0$.

$$f_Y(y) = \frac{1}{\sigma} f_X\left(\frac{y-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}, \quad y \in \mathbb{R}$$

We say Y has Normal distribution $N(\mu, \sigma^2)$ ($Y \overset{d}{=} N(\mu, \sigma^2)$)
Trivially $X \overset{d}{=} N(0,1)$ ← Standard Normal.

$\frac{1}{\sqrt{2\pi\sigma^2}}$



— Bell

(2) $X \overset{d}{=} EXP(\lambda)$. $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$.

$Y = aX$, $a > 0$.

$$f_Y(y) = \frac{\lambda}{a} e^{-\frac{\lambda}{a} x} , \quad x \geq 0$$

$$Y \overset{d}{=} EXP\left(\frac{\lambda}{a}\right).$$

Ex. Uniform & Cauchy. Try !

LEMMA 2 b.o.

Let X be a cts r.v. with pdf f & CDF F.

Define $Y := \frac{1}{a}(X-b)^2$ $a>0$, $b \in \mathbb{R}$.

Then Y is a cts r.v.

$$f_Y(y) = \frac{\sqrt{a}}{2\sqrt{y}}\left(f_X(b+\sqrt{ay}) + f_X(b-\sqrt{ay})\right), \quad y>0.$$

& at pts of discty on RHS.

Proof:

$$F_Y(y) = \mathbb{P}(Y \leq y) \quad , y > 0$$

$$= \mathbb{P}((X-b)^2 \leq ay)$$

$$= \mathbb{P}(-\sqrt{ay} \leq X - b \leq \sqrt{ay})$$

$$= \mathbb{P}(b - \sqrt{ay} \leq X \leq b + \sqrt{ay})$$

$$= F_X(b+\sqrt{ay}) - F_X(b-\sqrt{ay}) \quad \text{(since } X \text{ is cts)}$$

$\left[F_Y(y) = 0 \text{ if } Y \leq y \right]$

F_Y is a CDF & it is diff'ble except at fin. many points.

If $b+\sqrt{ay}$ & $b-\sqrt{ay}$ are cty points of f_X

(i.e., F_X is diff'ble at $b+\sqrt{ay}$, $b-\sqrt{ay}$)

then $$f_Y(y) = F_Y'(y) = \frac{d}{dy}\left(F_X(b+\sqrt{ay}) - F_X(b-\sqrt{ay})\right)$$

(use chain rule & lin.)

$$= \frac{1}{2}\sqrt{\frac{a}{y}}\left(f_X(b+\sqrt{ay}) + f_X(b-\sqrt{ay})\right).$$

Define $f_Y(y) = \frac{\sqrt{a}}{2\sqrt{y}}\left(f_X(b+\sqrt{ay}) + f_X(b-\sqrt{ay})\right)$, $y>0$. 0 Else.

Easily f_Y is p-cts & $\int_{-\infty}^{\infty} f_Y(y)\,dy = 1$..

& $F_Y(y) = \int_{-\infty}^{y} f_Y(y)\,dy$

So f_Y is a pdf for Y. ▨

Gen. n – See THM 7.1 of Ross.

Eg 26.7 $X \stackrel{d}{=} N(0,1)$ $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$Y = X^2$.

f_X is even

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right) = \frac{f_X(\sqrt{y})}{\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi y}} e^{-y/2} = \frac{y^{-1/2} e^{-y/2}}{\sqrt{2\pi}}$$

Y has Gamma $(1/2, 1/2)$ distribution. i.e, $Y \stackrel{d}{=} \Gamma(1/2, 1/2)$.

Since f_Y is a pdf $\int_0^{\infty} y^{-1/2} e^{-y/2} = \sqrt{2\pi}$.

$$\frac{1}{\sqrt{2}} \int_0^{\infty} y^{-1/2} e^{-y/2} = \sqrt{\pi}.$$

Eg 26.8 [Gamma(ν, λ) distribution] $\Gamma(\nu, \lambda)$, $\nu > 0$, $\lambda > 0$

ν – shape parameter, λ – scale parameter.

$X \stackrel{d}{=} \Gamma(\nu, \lambda)$ if $f_X(t) = \begin{cases} \frac{\lambda^{\nu} t^{\nu-1} e^{-\lambda t}}{\Gamma(\nu)} & t > 0 \\ 0 & t \leq 0 \end{cases}$

where $\Gamma(\nu) := \lambda^{\nu} \int_0^{\infty} t^{\nu-1} e^{-\lambda t}\, dt$. , Γ - Gamma
ν - nu

Check: f_X is cts except at $t=0$.

$\Gamma(\nu)$ - Gamma integral; $u = \lambda t$, $du = \lambda dt$

$$\Gamma(\nu) = \lambda^{\nu} \int_0^{\infty} t^{\nu-1} e^{-\lambda t}\, dt = \int_0^{\infty} u^{\nu-1} e^{-u}\, du.$$

$\Gamma(\nu)$ doesn't depend on λ.

f_X is well-defined if $0 < \Gamma(\nu) < \infty$

∵ Since $\nu > 0$, $\Gamma(\nu) > 0$.

Is $\Gamma(\nu) < \infty$? Integration by parts.

$$\Gamma(\nu+1) = \int_0^{\infty} u^{\nu} e^{-u}\, du = \left[-e^{-u} u^{\nu} \Big|_0^{\infty} + \nu \int_0^{\infty} u^{\nu-1} e^{-u}\, du \right]$$

$$= 0 + \nu\Gamma(\nu) \qquad \left(\lim_{u\to\infty} u^{\nu} e^{-u} = 0\right)$$

(since $\nu > 0$)

So $\Gamma(\nu+1) = \nu\Gamma(\nu)$.

So if $\Gamma(\nu) < \infty$ $\forall$ $\nu \in (0,1]$ then

$\Gamma(\nu) < \infty$ $\forall$ $\nu > 0$.

$$\Gamma(1) = \int_0^{\infty} e^{-u}\, du = 1.$$

$\Rightarrow$ $\Gamma(n+1) = n!$ $\forall$ $n \in \mathbb{N}$.

$$\Gamma(1/2) = \int_0^{\infty} u^{-1/2} e^{-u}\, du = \frac{1}{\sqrt{2}} \int_0^{\infty} u^{-1/2} e^{-u/2}\, du = \sqrt{\pi}$$

$\Rightarrow$ $\Gamma(n + 1/2) < \infty$ $\forall$ $n \in \mathbb{N}$.

Check $\Gamma(\nu) < \infty$ $\forall$ $\nu \in (0,1]$.