23/12 LECTURE 27 : EXP. OF CTS R.V.

X cts r.v. $\iff \exists$ a pdf f_X $\quad f_X : \mathbb{R} \to [0,\infty)$ p.cts, $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$\iff \exists$ a cts DF $F_X : \mathbb{R} \to [0,1]$ $\ni$ F_X' exists except at fin. many pts.

$$\mathbb{P}(X \in (a,b]) = \int_a^b f_X(x)\,dx = F_X(b) - F_X(a) \qquad -\infty \le a < b < \infty.$$

THM: 27.1 Let X be a cts r.v. $\ni$ F_X is a strictly $\uparrow$ cts fn.

Then (1) $F_X^{-1}(U) \stackrel{d}{=} X$ $\quad$ ($U \stackrel{d}{=} U(0,1)$ & F_X^{-1} is well-defined on $(0,1)$)

(2) $F_X(X) \stackrel{d}{=} U$.

Proof: (1) F_X is strictly $\uparrow$ & so F_X^{-1} is well-defined on $(0,1)$ & also strictly increasing (Ex.)

$x \in \mathbb{R}$, $\mathbb{P}(F_X^{-1}(U) \le x) \stackrel{\Downarrow}{=} \mathbb{P}(U \le F_X(x)) = F_X(x)$.

So X & $F_X^{-1}(U)$ have same CDFs i.e., $X \stackrel{d}{=} F_X^{-1}(U)$
(equal in distribution).

(2) Similarly, $x \in (0,1)$.

$$\mathbb{P}(F_X(X) \le x) \underset{F_X \text{ strictly } \uparrow}{=} \mathbb{P}(X \le F_X^{-1}(x)) = F_X(F_X^{-1}(x))$$

$$\underset{F_X \text{ is strictly } \uparrow}{=} x = \mathbb{P}(U \le x)$$

$\Rightarrow F_X(X) \stackrel{d}{=} U$. $\quad$ ∎

Applications: output of RAND is U. To generate X, you take $F_X^{-1}(\text{RAND})$.

Qn: Suppose X is a discrete r.v. (For eg. X has pmf p on $\{1,..,n\}$)
Is there $h : \mathbb{R} \to \mathbb{R}$ $\ni$ $h(U) \stackrel{d}{=} X$?

Defn 27.2 X be cts r.v. We define mean / Expectation of X as
$$E[X] := \int_{-\infty}^{\infty} x f_X(x)\,dx \quad \text{if} \quad \int_{-\infty}^{\infty} |x| f_X(x)\,dx < \infty.$$
Else undefined or $E[X]$ doesn't exist.

[X Discrete, $E[X] = \sum_x x\, p_X(x)$ if $\sum_x |x| p_X(x) < \infty$]

Note $x \mapsto x f_X(x)$ is p.cts.

Eg 27.3 (1) $U \stackrel{d}{=} U(0,1)$. $E[X] = \int_0^1 x\,dx = \frac{1}{2}$.

[Check $\int_{-\infty}^{\infty} |x| f_X(x)\,dx = \int_0^1 x\,dx < \infty$]

(2) $X \stackrel{d}{=} EXP(\lambda)$ $E(X) = \lambda \int_0^{\infty} x e^{-\lambda x}\,dx \underset{\text{IBP}}{=} \frac{1}{\lambda}$.

Check $E[X]$ exists

(3) $X \stackrel{d}{=} N(0,1)$ Check $\int_{-\infty}^{\infty} |x| f_X(x)\,dx < \infty$.

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

[$\underset{f_X \text{ even}}{=} 2\int_0^{\infty} x f_X(x)\,dx$; $f_X(x) \le \frac{C}{x^3}$ $\forall x > 0$]

$$= \int_0^{\infty} x f_X(x)\,dx + \int_{-\infty}^{0} x f_X(x)\,dx$$

$$= \int_0^{\infty} x f_X(x)\,dx + \int_{\infty}^{0} x f_X(-x)\,dx$$

(CoV $x \mapsto -x$)

$$= \int_0^{\infty} x f_X(x)\,dx - \int_0^{\infty} x f_X(-x)\,dx$$

$$= \int_0^{\infty} x f_X(x)\,dx - \int_0^{\infty} x f_X(x)\,dx$$

(since f_X is even i.e, $f_X(x) = f_X(-x)$).

$$= 0.$$

- If f_X is even & $\int_0^{\infty} x f_X(x)\,dx < \infty$ then $\mathbb{E}[X] = 0$.
  (Same proof as above)
- If $\int_{-\infty}^{\infty} |x| f_X(x)\,dx < \infty$ then $\lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b x f_X(x)\,dx = \int_{-\infty}^{\infty} x f_X(x)\,dx$

If $\sum_{n=1}^{\infty} |a_n| < \infty$ then $\lim_{n \to \infty} \sum_{i=1}^{n} a_n$ exists $\quad (a_n = (-1)^n,\ n \geq 0)$

Eg 27.4 $X \overset{d}{=}$ Cauchy $\quad f_X(x) = \dfrac{1}{\pi} \dfrac{1}{(1+x^2)}$

f_X is even. $\int_{-\infty}^{\infty} |x| f_X(x)\,dx = \dfrac{2}{\pi} \int_0^{\infty} \dfrac{x}{1+x^2}\,dx = \infty$! check.

So $\mathbb{E}[X]$ doesn't exist.

(Compare with $\sum_{n=1}^{\infty} \dfrac{n}{1+n^2}$)

Defn 27.5 Let X be cts r.v. & $h : \mathbb{R} \to \mathbb{R}$ p-cts.

$$\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x)\,dx \quad \text{if } \int_{-\infty}^{\infty} |h(x)| f_X(x)\,dx < \infty.$$

else undefined / $\mathbb{E}[h(X)]$ doesn't exist.

VARIANCE. $\mathrm{VAR}[X] := \mathbb{E}[(X - \mu_X)^2]$ if $\mathbb{E}X^2$ exists

where $\mu_X := \mathbb{E}[X]$.

k^{th} MOMENT – $\mathbb{E}[X^k]$, $k = 1, 2, \ldots$ if it exists.

THM 27.6 (1) If $\mathbb{E}[X]$ exists then $\forall\, a, b \in \mathbb{R}$

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

(2) If $h_1 \leq h_2$ and $\mathbb{E}[h_1(X)]$ exists

then $\mathbb{E}[h_1(X)] \leq \mathbb{E}[h_2(X)]$ (Monotonicity)

(3) If $\mathbb{E}[X^2]$ exists

$$\mathrm{VAR}[X] = \mathbb{E}[X^2] - \mu_X^2$$

(4) If $\mathbb{E}[X^2]$ exists, then $\forall a, b \in \mathbb{R}$.

$$\mathrm{VAR}[aX+b] = a^2 \mathrm{VAR}[X]$$

Proof: (1) To check $\int_{-\infty}^{\infty} |ax+b| f_X(x)\,dx < \infty$.

observe $\quad |ax+b| \le |a||x| + |b|$. ('triangle ineq.')

$$\int_{-\infty}^{\infty} |ax+b| f_X(x)\,dx \le \int_{-\infty}^{\infty} (|a||x| + |b|) f_X(x)\,dx$$

(mon. for non-neg. integrals)

(linearity of integrals) $\le |a| \int_{-\infty}^{\infty} |x| f_X(x)\,dx + |b| \int_{-\infty}^{\infty} f_X(x)\,dx$

since $\mathbb{E}[X]$ exists $\quad < \infty \quad + |b| < \infty$.

f_X is pdf

so $\mathbb{E}[aX+b]$ exists.

by defn $\quad \mathbb{E}[aX+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\,dx$

(lin. of integrals) $= a\int_{-\infty}^{\infty} x f_X(x)\,dx + b\int_{-\infty}^{\infty} f_X(x)\,dx$

(defn of $\mathbb{E}[X]$ & f_X is a pdf) $= a\,\mathbb{E}[X] + b$

(2) Ex.

(3) $\mathrm{VAR}[X] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx$ (defn)

$= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - 2\mu_X \int_{-\infty}^{\infty} x f_X(x)\,dx + \mu_X^2 \int_{-\infty}^{\infty} f_X(x)\,dx$ (linearity of integral)

$= \mathbb{E}[X^2] - 2\mu_X^2 + \mu_X^2$ (by defns)

$$= \mathbb{E}[X^2] - \mu_X^2 .$$

(4) Let $Y = aX + b$.

From (1), $\mu_Y = a\mu_X + b$

$$\int_{-\infty}^{\infty} (ax+b)^2 f_X(x)\,dx \leq a^2 \int_{-\infty}^{\infty} x^2 f_X(x)\,dx + 2|b| \int_{-\infty}^{\infty} |x| f_X(x)\,dx + b^2$$

$< \infty$ as $\mathbb{E}[X^2]$, $\mathbb{E}[X]$ exist.

so $\mathbb{E}[(aX+b)^2]$ exists.

$$\Rightarrow \quad \mathrm{VAR}[aX+b] = \int_{-\infty}^{\infty} (ax+b - a\mu_X - b)^2 f_X(x)\,dx$$

(defn & $\mu_Y = a\mu_X + b$)

$$= a^2 \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx = a^2\,\mathrm{VAR}[X]. \quad \blacksquare$$

Note: $\quad Y = aX + b \qquad f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$

$$\mathbb{E}[aX+b] = \mathbb{E}[Y]$$

(1) of thm $\rightarrow$ $\| $ $a\int_{-\infty}^{\infty} x f_X(x)\,dx + b.$

$$= \int_{-\infty}^{\infty} y f_Y(y)\,dy = \frac{1}{a}\int_{-\infty}^{\infty} y f_X\left(\frac{y-b}{a}\right) dy$$

$$= \int_{-\infty}^{\infty} (ax+b) f_X(x)\,dx.$$

(Change $\frac{y-b}{a} \rightarrow x$; $dy = a\,dx$)

So defn. of $\mathbb{E}[h(X)]$ is consistent in the simple case of $h(X) = aX + b$!!!

[For more gen. case. Say $Y = h(X)$ then $\mathbb{E}[Y] = \mathbb{E}[h(X)]$ follows by a gen. CoV]

Eg 27.7

(1) U. $\mathbb{E}[U] = \frac{1}{2}$.

$k \geq 1$ $\mathbb{E}[U^k] = \int_0^1 x^k \, dx = \frac{1}{k+1}$.

$\mathrm{VAR}[U] = \mathbb{E}[U^2] - (\mathbb{E}[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

(2) $X \stackrel{d}{=} U(b, b+a)$ $f_X(x) = \begin{cases} \frac{1}{a} & x \in (b, b+a) \\ 0 & \text{elsewhere} \end{cases}$, $a > 0$. $b \in \mathbb{R}$.

From previous class.

$X \stackrel{d}{=} Y$, $Y = \underline{aU} + b$.

$\mathbb{E}[X] = \mathbb{E}[Y] \underset{\text{Thm 27.6(i)}}{=} a\,\mathbb{E}[U] + b = \frac{a}{2} + b$.

$\mathrm{VAR}[X] \overset{\text{Thm 27.6(4)}}{=} a^2 \, \mathrm{VAR}[U] = \frac{a^2}{12}$.

(3) $X \stackrel{d}{=} \mathrm{EXP}(\lambda)$ $\mathbb{E}[X] = \frac{1}{\lambda}$

$\mathbb{E}[X^2] = \lambda \int_0^\infty x^2 e^{-\lambda x} \, dx = \frac{2}{\lambda^2}$

$\mathrm{VAR}[X] = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$.

(4) $Z \stackrel{d}{=} N(0,1)$ $X \stackrel{d}{=} N(\mu, \sigma)$ $\sigma > 0$.

Last class, if $Y = \sigma Z + \mu$, then $Y \stackrel{d}{=} X$.

$\mathbb{E}[X] = \mathbb{E}[Y] = \sigma \underbrace{\mathbb{E}[Z]}_{0} + \mu = \mu$.

$\mathrm{VAR}[X] = \sigma^2 \mathrm{VAR}[Z]$.

$\mathrm{VAR}[Z] = \mathbb{E}[Z^2]$ $(\mathbb{E}[Z] = 0)$

$= \int_{-\infty}^{\infty} z^2 f_Z(z) \, dz$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty z^2 e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{1/2} e^{-x/2}\, dx \quad \leftarrow x = z^2 \quad dz = \frac{dx}{2\sqrt{x}}$$

$$\overset{(?)}{=} \frac{1}{\sqrt{2\pi}} \Gamma(3/2) = 1.$$

So $VAR[X] = \sigma^2 VAR[Z] = \sigma^2$.

$X \overset{d}{=} N(\mu, \sigma)$; $\mu = \mathbb{E}[X]$, $\sigma^2 = VAR[X]$

$\sigma = SD[X] = \sqrt{VAR[X]}$.

Exercises compute $\mathbb{E}[X^k]$ for EXP., $N(\mu,\sigma)$, . . .

Recall $X_n \overset{d}{=} Bin(n, \frac{\lambda}{n})$ $\lambda > 0$. $\mathbb{E}X_n = \lambda$, $Var[X_n] = \lambda(1-\frac{\lambda}{n})$

$k=0,1,\ldots$ $\mathbb{P}(X_n = k) \longrightarrow \mathbb{P}(X=k)$ $X \overset{d}{=} Poi(\lambda)$

↳ Poisson limit theorem / law of small numbers

"David Hand"!

$S_n \overset{d}{=} Bin(n,p)$ $p \in (0,1]$

$\mathbb{E}[S_n] = np$ $VAR[S_n] = np(1-p)$.

$a, b \in \mathbb{R}$

$$\mathbb{P}\left(a < \frac{S_n - np}{\sqrt{np(1-p)}} < b\right) \xrightarrow{d} \mathbb{P}\left(\underset{\overset{\parallel d}{N(0,1)}}{N} \in (a,b)\right)$$

De-Moivre Laplace Central limit theorem!

$\boxed{\text{Qns:}}$ (1) Given a seq r.v. X_n, $\mathbb{E}[X_n] \to \lambda$, $VAR[X_n] \to \lambda$

can one show Poisson limit theorem?

(2) Given a seq r.v. S_n, $\mathbb{E}[S_n] \to \infty$, $VAR[S_n] \to \infty$

is it true $\mathbb{P}\left(a < \frac{S_n - \mathbb{E}[S_n]}{\sqrt{VAR[S_n]}} < b\right) \to \mathbb{P}(N \in (a,b))$?

(CLT)

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THANK U