30/9 LECTURE 3: EXAMPLES OF PROB. SPACES.

Defn 3.1 Let Ω be a finite set. Let $p: \Omega \to [0,1]$ be a function such that (s.t.) $\sum_{\omega \in \Omega} p(\omega) = 1$. We call (Ω, p) a finite prob. space.

p is called prob. mass function (pmf).

Define $\mathbb{P}: 2^{\Omega} \to [0,1]$ as $\mathbb{P}(A) := \sum_{\omega \in A} p(\omega)$, $A \subseteq \Omega$ (subset of), $\mathbb{P}(\phi) := 0$.

2^{Ω} = Power-set of Ω = $\{A : A \subseteq \Omega\}$

Then $\mathbb{P}$ is called prob. distribution.

[Note: $|2^{\Omega}| = 2^{|\Omega|}$]

Find interesting (Ω, p) & an event A and compute $\mathbb{P}(A)$.

PROPN 3.2: [PROPOSITION] Let (Ω, p) be a finite prob. space and $\mathbb{P}$ as defined above. Then $\mathbb{P}$ satisfies the following properties.

(i) $\mathbb{P}(A) \geq 0 \quad \forall A \subseteq \Omega$ (ii) $\mathbb{P}(\Omega) = 1$

(iii) $\mathbb{P}(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i)$ if $A_1, \ldots, A_n \subseteq \Omega$ & are pairwise disjoint ($A_i \cap A_j = \phi \ \forall i \neq j$)

[finite additivity]

Proof: (i) Since $p(\omega) \geq 0$ then so is $\mathbb{P}(A) = \sum_{\omega \in A} p(\omega)$, $A \subseteq \Omega$.

(ii) $\mathbb{P}(\Omega) = \sum_{\omega \in \Omega} p(\omega) = 1$.

(by defn of $\mathbb{P}$) (by defn. of pmf)

(iii) We shall prove by induction on n.

finite additivity is trivially true for $n=1$ $(\mathbb{P}(A_1) = \mathbb{P}(A_1))$

Suppose it is true for $n-1$.

Let us consider f.d. for n.

Say $A_1, \ldots, A_n$ are pairwise disjoint.

Define $B := \bigcup_{i=1}^{n-1} A_i$.

Since $A_1, \ldots, A_n$ are pairwise disjoint then $A_n \cap B = \phi$.

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \mathbb{P}(A_n \cup B) \quad (\text{by defn of } B)$$

$$(A_1 \cup \cdots \cup A_n) \qquad = \sum_{\omega \in A_n \cup B} p(\omega) \quad (\text{defn of } \mathbb{P})$$

$$\left(\sum_{i \in I \cup J} a_i = \sum_{i \in I} a_i + \sum_{i \in J} a_i \quad I \cap J = \phi\right) \quad = \sum_{\omega \in A_n} p(\omega) + \sum_{\omega \in B} p(\omega) \quad (A_n \cap B = \phi)$$

$$= \mathbb{P}(A_n) + \mathbb{P}(B) \quad (\text{defn of } \mathbb{P})$$

$$= \mathbb{P}(A_n) + \sum_{i=1}^{n-1} \mathbb{P}(A_i) \quad (\text{since f.a. holds for } n-1)$$

$$= \sum_{i=1}^{n} \mathbb{P}(A_i).$$

■ — Q.E.D.

Ex. 3.3 Suppose $\mathbb{P} : 2^{\Omega} \to \mathbb{R}$ satisfying (i), (ii) & (iii) in PROPN 3.2.

Show that $\exists$ a $p : \Omega \to [0,1]$ s.t. (Ω, p) is a finite prob. space &

$\mathbb{P}$ is the prob. distribution corresponding to pmf p.

i.e., $\mathbb{P}(A) = \sum_{\omega \in A} p(\omega) \quad \forall A \subseteq \Omega$.

Given (Ω, p), we get $\mathbb{P}$ satisfying (i), (ii) & (iii). [Propn 3.2]

Given Ω & $\mathbb{P}$ [satisfying (i), (ii) & (iii)], we get that $\exists\, p \ni (\Omega, p)$ is a prob. space & $\mathbb{P}$ is the prob. distribution. [Ex. 3.3]

Ω finite set, $\Omega \neq \phi$.

Eg 3.4: $p(\omega) = \dfrac{1}{|\Omega|}$ [Uniform prob. distribution on Ω] (PD)

Easily check p is a pmf on Ω.

We'll consider uniform PD on the examples of Ω's we have discussed.

Eg. 3.5: $\Omega = \{0,1\}$. Define $p(0) = \frac{1}{2} = p(1)$. [Tossing a fair coin] Since prob. of H = prob. of T

Easy to compute $\mathbb{P}(A)$ $\forall$ $A \subseteq \Omega$.

$A = \phi \Rightarrow \mathbb{P}(A) = 0$; $A = \{0\}$, $\mathbb{P}(A) = p(0) = 1/2$

$A = \{1\} \Rightarrow \mathbb{P}(A) = p(1) = 1/2$; $A = \Omega \Rightarrow \mathbb{P}(\Omega) = p(0) + p(1)$
$= 1$.

Eg 3.6 $\Omega = \{0,1\}^2 = \{(0,0), (0,1), (1,0), (1,1)\}$ [Tossing two coins]

$p((0,0)) = p((0,1)) = p((1,0)) = p((1,1)) = \frac{1}{4}$. $\leftarrow$ [both coins are fair]

$\mathbb{P}$ - uniform PD.

A = first toss is a head $= \{(1,0), (1,1)\}$

$\mathbb{P}(A) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

B = Two tosses are not the same $= \{(0,1), (1,0)\}$

$\mathbb{P}(B) = 1/2$.

Ex. 3.7: (Ω, p) is prob. space with uniform PD. Then show that

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}.$$

Eg 3.8: $\Omega = \{1, \ldots, 6\}$ (Roll a dice).

$p(\omega) = \frac{1}{6}$ (fair dice)

A = Dice shows an even number $= \{2,4,6\}$

So $\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{1}{2}$

(By Ex.)

A = Dice shows a multiple of $3 = \{3,6\}$

$\mathbb{P}(A) = \frac{1}{3}$

Eg 3.9: $\Omega = \{0,1\}^n = \{(a_1, \ldots, a_n) : a_i \in \{0,1\}\ \forall\, 1 \le i \le n\}$ [bit string or n coin tosses]

$p(\omega) = \frac{1}{2^n}$ $\omega = (\omega_1, \dots, \omega_n) \in \Omega.$ [Toss coins randomly or message is chosen randomly]

B_k = set of bit strings with exactly k 1's (or H's)

$= \{\omega : \sum_{i=1}^{n} \omega_i = k\}$, $k = 0, \dots, n.$

$\mathbb{P}(B_k) = \frac{|B_k|}{|\Omega|} = \frac{|B_k|}{2^n} = \binom{n}{k}\frac{1}{2^n}$ $\left[|B_k| \underset{\text{Ex.}}{=} \binom{n}{k} \right]$

Eg 3.10: $\Omega = [52]$; $p(\omega) = \frac{1}{52}$, $\omega \in \Omega.$ [Pick a card uniformly at random (uar)]

A_s = set of spades

$\mathbb{P}(A_s) = \frac{|A_s|}{|\Omega|} = \frac{1}{4}.$

A_1 = set of 1's $\Rightarrow P(A_1) = \frac{|A_1|}{|\Omega|} = \frac{1}{13}.$

Space of all outcomes — Ω

how an outcome is chosen in the experiment — p or $\mathbb{P}$

Event – colln. of outcomes, $A \subseteq \Omega.$

In prev. example, Prob. a card selected uar is a spade $= \mathbb{P}(A_s).$

Eg. 3.11: [BOSE-EINSTEIN PROB. DISTRIBUTION]

$\Omega_{r,n} = \{(r_1, \dots, r_n) : r_i \geq 0 \ \forall i \ \& \ \sum_{i=1}^{n} r_i = r\}$ [Configuration of unlabelled particles]

$p(\omega) = \frac{1}{|\Omega_{r,n}|}$, $\omega \in \Omega_{r,n}.$

A_i = ith cell is empty. $\mathbb{P}(A_i) = ?$

Eg 3.12: [Fermi - Dirac PD] [Configuration of electrons (unlabelled & not two particles in the same site)]

$\Omega^*_{r,n} = \{ (r_1, \ldots, r_n) : r_i \in \{0,1\} \ \forall i \ \& \ \sum_{i=1}^{n} r_i = r \}$

$P(\omega) = \frac{1}{|\Omega^*_{r,n}|}$

A_i = i^{th} cell is empty. $P(A_i) = ?$

ASIDE: Perform an experiment n times. Observe the outcomes & see how many outcomes are in $A_1, \ldots, A_k$ $(A_1, \ldots, A_k \subseteq \Omega)$

If $\forall i$ $P(A_i) \approx \frac{\text{\# of times outcome in } A_i}{n}$ then the prob. model is "good".

Is the coin really fair?

Toss a coin n times. If $\frac{\text{\# of heads}}{n} \approx \frac{1}{2}$ then Yes.

$\frac{1}{2}$ ‖ Prob. of heads

For eg. $\frac{\text{\# of heads}}{n} = \underline{0.55}$

$n = 100$