5/10 LECTURE 4: MORE EXAMPLES & CALCULATIONS

→ We'll refer to prob. spaces as (Ω, p) or $(\Omega, \mathbb{P})$ as specifying pmf or PD are equivalent.

→ If $\mathbb{P} : 2^{\Omega} \to [0, \infty)$ satisfies $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ for $A \cap B = \phi$ then it also satisfies finite additivity. [Ex.] $\left[2^{\Omega} = \{A : A \subseteq \Omega\} \quad \wp(\Omega) \right]$

$\boxed{\text{THM } 5.1}$ Let $(\Omega, \mathbb{P})$ be a fin. prob. space. Then the following hold.

(i) $\mathbb{P}(A) \leq \mathbb{P}(B) \quad \forall \ A \subseteq B \subseteq \Omega.$ [Monotonicity]

(ii) $\mathbb{P}(A) \leq 1 \quad \forall \ A \subseteq \Omega$

(iii) $\mathbb{P}(\phi) = 0$

(iv) $\mathbb{P}(A_1 \cup \dots \cup A_n) \leq \sum_{i=1}^{n} \mathbb{P}(A_i) \quad \forall \ A_1, \dots, A_n \subseteq \Omega.$ [finite subadditivity]

(v) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ [Inclusion-Exclusion]

Proof later.

Remarks: (i) $\Rightarrow$ (ii) is easy. Take $B = \Omega$ & use $\mathbb{P}(\Omega) = 1$.

(iii) follows because $\mathbb{P}(A \cup \phi) = \mathbb{P}(A) + \mathbb{P}(\phi)$ for any $A \subseteq \Omega$. as $A \cap \phi = \phi$.

$\Rightarrow \mathbb{P}(A) = \mathbb{P}(A) + \mathbb{P}(\phi)$

$\Rightarrow \mathbb{P}(\phi) = 0.$

Eg 5.2 Let $(\Omega, \mathbb{P})$ be unif. prob. space i.e., $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$.

Then (i)–(v) can be re-stated respectively as

(i) $|A| \leq |B|$ for $A \subseteq B \subseteq \Omega$ $\quad$ [$\Leftrightarrow$ $\mathbb{P}(A) \leq \mathbb{P}(B)$] Equivalent to

(ii) $|A| \leq |\Omega|$ for $A \subseteq \Omega$. $\quad$ [$\Leftrightarrow$ $\mathbb{P}(A) \leq 1$]

(iii) $|\phi| = 0$ $\quad$ [$\Leftrightarrow$ $\mathbb{P}(\phi) = 0$]

(iv) $|A_1 \cup \dots \cup A_n| \leq \sum_{i=1}^{n} |A_i| \quad \forall \ A_1, \dots, A_n \subseteq \Omega$ [$\dots$

(v) $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ — Proof via Venn diagrams.

Eg 5.3: (Bose-Einstein prob. distribution) n cells, r unlabelled particles.

$$\Omega_{r,n} = \{(r_1, \dots, r_n) : r_i \geq 0 \ \forall i \ \& \ \sum_{i=1}^{n} r_i = r\}.$$

A_i = i^{th} cell is empty $= \{(r_1, \dots, r_n) \in \Omega_{r,n} : r_i = 0\}$

$\mathbb{P}$ - uniform PD i.e.,
$$\mathbb{P}(A) = \frac{|A|}{|\Omega_{r,n}|}.$$

Compute $\mathbb{P}(A_i)$.

By defn of $\mathbb{P}$, $\mathbb{P}(A_i) = \dfrac{|A_i|}{|\Omega_{r,n}|}$.

$$\begin{aligned} A_i &= \{(r_1, \dots, r_n) \in \Omega_{r,n} : r_i = 0\} \\ &= \{(r_1, \dots, r_n) : r_j \geq 0 \ \forall j, \ \sum_{j=1}^{n} r_j = r, \ r_i = 0\} \\ &= \{(r_1, \dots, r_n) : r_j \geq 0 \ \forall j, \ \sum_{\substack{j=1 \\ j \neq i}}^{n} r_j = r, \ r_i = 0\} \\ &= \{(r_1, \dots, r_n) \in \Omega_{r,n} : \sum_{\substack{j=1 \\ j \neq i}}^{n} r_j = r\} \end{aligned}$$

$$B = \{(r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n) : r_j \geq 0 \ \forall j \neq i, \ \sum_{\substack{j=1 \\ j \neq i}}^{n} r_j = r\}$$

$|B| = |\Omega_{r,n-1}| \Longleftarrow$ r unlabelled particles in $(n-1)$ cells.

(Ex.) $\exists$ a bijection from A_i to B

[Hint: $(r_1, \dots, r_n) \in A_i \xmapsto{\text{maps to}} (r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n) \in B$]

So $|A_i| = |B| = |\Omega_{r,n-1}|$

$$\Rightarrow \quad \mathbb{P}(A_i) = \frac{|\Omega_{r,n-1}|}{|\Omega_{r,n}|}.$$

[$|\Omega_{r,n}|$ in Assignment] substitute & find the value.

Suppose A = at least one cell is empty.

Then $A = \bigcup_{i=1}^{n} A_i$. But A_i's aren't disjoint.

finite sub-add $\Rightarrow$ $\mathbb{P}(A) \leq \sum_{i=1}^{n} \mathbb{P}(A_i) = \dfrac{n|\Omega_{r,n-1}|}{|\Omega_{r,n}|}$

If $\dfrac{n\,|\Omega_{r,n-1}|}{|\Omega_{r,n}|} \geq 1$ then the upper bound is "useless" as we know that $\mathbb{P}(A) \leq 1$. $\square$

PROPERTY of $\mathbb{P}$.

LEMMA 5.4: Let $(\Omega, \mathbb{P})$ be prob. space. Then $\forall\ A \subseteq B$

$$\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A).$$

In particular $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

Proof: $B = \underbrace{A \sqcup (B \setminus A)}$ $\rightarrow$ disjoint union i.e., $A \cap (B \setminus A) = \phi$ (union of disjoint sets)

finite add. $\Rightarrow$ $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A)$

$\Rightarrow$ $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$.

If $B = \Omega$ then $B \setminus A = A^c$ & $\mathbb{P}(B) = 1$.

This gives $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$. $\blacksquare$

Back to Bose-Einstein PD.

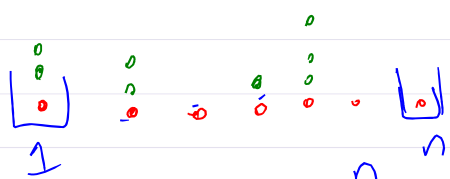
$$A = \text{at least 1 cell is empty} = \bigcup_{i=1}^{n} A_i$$

$$A^c = \bigcap_{i=1}^{n} A_i^c = \{(r_1, \dots, r_n) \in \Omega_{r,n} : r_i \geq 1\ \forall i\}$$

$$= \{(r_1, \dots, r_n) : r_i \geq 1\ \forall i,\ \sum_{i=1}^{n} r_i = r\}$$

Trivially $A^c = \phi$ if $r < n$

$\Rightarrow$ $\mathbb{P}(A) = 1$ if $r < n$.



Assume $r \geq n$. $A^c = \{(r_1, \dots, r_n) : r_i \geq 1\ \forall i,\ \sum_{i=1}^{n} r_i = r\}$

[Ex: show this is a bijection] $\longleftarrow$ $\Big\downarrow$ – maps to

$A^c \to \Omega_{r-n,n}$ is a bijection.

$$\{(r_1 - 1, \dots, r_n - 1) : r_i - 1 \geq 0\ \forall i,\ \sum_{i=1}^{n} (r_i - 1) = r - n\}$$

$\parallel$ (by putting $s_i = r_i - 1$)

$$\Omega_{r-n,n} = \{(s_1, \dots, s_n) : s_i \geq 0\ \forall i,\ \sum_{i=1}^{n} s_i = r - n\}$$

$r-n$ unlabelled particles into n cells.

So $|A^c| = |\Omega_{r-n,n}|$.

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{|A^c|}{|\Omega_{r,n}|} = 1 - \frac{|\Omega_{r-n,n}|}{|\Omega_{r,n}|},$$

Substitute values for $|\Omega_{r-n,n}|$ & $|\Omega_{r,n}|$ from Assignment. if $r \geq n$.

Eg: A = At least one cell has at least 2 particles. $\mathbb{P}(A) = ?$

Eg 6.5: (Fermi-Dirac distribution). Assume $r \leq n$.

$$\Omega^*_{r,n} = \{(r_1, \dots, r_n) : r_i \in \{0,1\} \ \forall i \ \& \ \sum_{i=1}^{n} r_i = r\}$$

A_i = i^{th} cell is empty. [if $r > n$, $\Omega^*_{r,n} = \phi$]

Compute as before $\mathbb{P}(A_i) = \frac{|A_i|}{|\Omega^*_{r,n}|}$ — (>0 as $r \leq n$)

Show like in BE PD, $|A_i| = |\Omega^*_{r,n-1}|$

i.e., if i^{th} cell is empty, we are distributing r particles in $(n-1)$-cells & no two particles in a cell.

$$\Rightarrow \mathbb{P}(A_i) = \frac{|\Omega^*_{r,n-1}|}{|\Omega^*_{r,n}|}$$ (substitute values from Assignment)

$\to$ (>0 because $r \leq n$)

[Note $r = n \Rightarrow \Omega^*_{r,n-1} = \phi \Rightarrow \mathbb{P}(A_i) = 0$. Easy]

$A = \bigcup_{i=1}^{n} A_i$ = at least one cell is empty.

$$\mathbb{P}(A) = \begin{cases} 1 & r < n \\ 0 & r = n \end{cases}$$