12/10 LECTURE 6 — MORE EXAMPLES.

Eg 6.1 (Sampling without replacement) $k \le n$.

Choose k objects from n objects without repetition/replacement done by-one.

Ω = set of all ordered k-tuples from $[n]$ with no repetition

$= \{ \omega = (\omega_1, \dots, \omega_k) \in [n]^k : \omega_i \neq \omega_j \ \forall i \neq j \}$

$|\Omega| = n(n-1) \cdots (n-k+1) = \dfrac{n!}{(n-k)!}$

ω_i — type 1 if $\omega_i \le m$, $1 \le m \le n$.

ω_i — type 0 if $\omega_i > m$

A = $\exists$ are exactly l objects of type 1

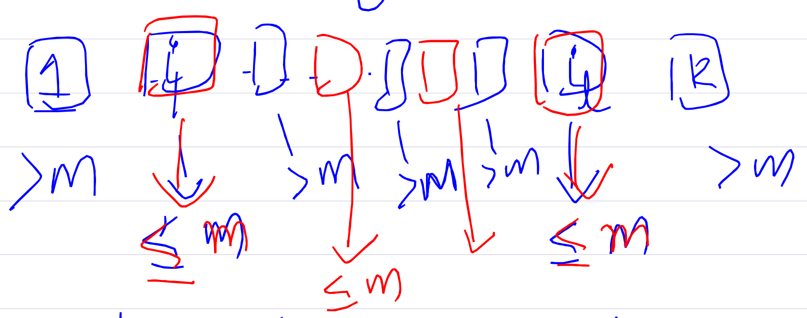
$= \{ \omega \in \Omega : \#\{ i : \omega_i \le m \} = l \}$, $l \le k$, $l \le m$

Assume k objects are chosen UAR. What is $\mathbb{P}(A) = ?$

Concrete eg: $1, \dots, n$ are prices of objects in a shop.

k customers come & buy one object each UAR.

A = Exactly l customers bought objects of price $\le m$.



$i_1, \dots, i_l$ distinct positions in $1, \dots, k$

There are l choices in $1, \dots, k$ for $\omega_i \le m$ & remaining $k-l$ choices, $\omega_i > m$. Choose $i_1, \dots, i_l \in [k]$ & distinct.

$A(i_1, \dots, i_l) = \{ \omega \in \Omega : \omega_{i_1}, \dots, \omega_{i_l} \le m$ & $\omega_j > m \ \forall j \neq i_1, \dots, i_l \}$.

$$A = \bigsqcup_{\substack{i_1, \dots, i_l \le k \\ \text{distinct}}} A(i_1, \dots, i_l)$$

$$\mathbb{P}(A) = \sum_{\substack{i_1,\dots,i_\ell \le k \\ \text{distinct}}} \mathbb{P}(A(i_1,\dots,i_\ell)) \quad (\text{fin- add-})$$

$$= \sum_{\substack{i_1,\dots,i_\ell \le k \\ \text{distinct}}} \frac{|A(i_1,\dots,i_\ell)|}{|\Omega|} \quad (\text{UAR})$$

$$|A(i_1,\dots,i_\ell)| = \underbrace{m(m-1)\cdots(m-\ell+1)}_{\substack{\text{fill } \ell \text{ positions with} \\ 1,\dots,m \text{ without repetition}}} \underbrace{(n-m)(n-m-1)\cdots(n-m-(k-\ell)+1)}_{\substack{\text{fill } (k-\ell) \text{ positions} \\ \text{with } m+1,\dots,n \\ \text{without repetition}}}$$

so $$\mathbb{P}(A) = \binom{k}{\ell} \frac{m!}{(m-\ell)!} \frac{(n-m)!}{(n-m-(k-\ell))!} \Big/ \frac{n!}{(n-k)!}$$

$$= \frac{\binom{m}{\ell}\binom{n-m}{k-\ell}}{\binom{n}{k}}.$$

Ex 6.2 Choose $S \subseteq [n]$, $|S| = m$. $A_S = \{w \in \Omega : \#\{i : w_i \in S\} = \ell\}$

Show that $\mathbb{P}(A_S) = \dfrac{\binom{m}{\ell}\binom{n-m}{k-\ell}}{\binom{n}{k}}$ n, m, k, l same as in Eg 6.1

Ex 6.3. Assume k (unordered) objects are chosen UAR from n objects. Compute $\mathbb{P}(A_S)$ for the same.

There are m objects of type 1 in n objects. k obj. are chosen uniformly at random from n objects in order.

$\mathbb{P}(\exists$ exactly l objects of type 1 in k objects$) = \dfrac{\binom{m}{l}\binom{n-m}{k-l}}{\binom{n}{k}}$

Eg 6.4 A factory has m defective watches out of n. k watches are selected UAR & each watch is checked. l of them are found defective. Estimate m.

$p(k,l) = \mathbb{P}($ exactly l defective watches in k selected$) = \dfrac{\binom{m}{l}\binom{n-m}{k-l}}{\binom{n}{k}}$

(from Eg 6.1)

Estimate "statistically" $p(k,l)$ & then find m.

"Repeat experiment (select k UAR) & compute proportion of experiments with exactly l defective watches."

Eg 6.5: There are n fish in a lake (unknown)

Ecologists catch m fish & then mark them RED & put them back. After some time, you catch r fish UAR & k red fish are observed.

$\mathbb{P}(k$ red fish in r fish$) = \dfrac{\binom{m}{k}\binom{n-m}{r-k}}{\binom{n}{r}}$

- Technique for estimating population size 'n'.

Why not take $\left(\dfrac{k}{r} = \dfrac{m}{n}\right)$?