14/10 LECTURE 7 : MORE EXAMPLES — NON-UNIFORM PROB. SPACES.

Eg 7.1 : A standard deck of cards is "well-shuffled" (= cards are arranged UAR). What is Prob. first 10 cards have a set of 4 aces?

Sample space Ω = possible arrangements of 52 cards

$= \{(a_1, \ldots, a_{52}) : a_i \in [52], a_i \neq a_j \ \forall i \neq j\}$

(a_i — the card in the i^{th} position i.e., i^{th} card)

(Sampling without replacement 52 objects $(k=)$ from 52 objects $(n=)$)

$k=n$ in sampling without replacement is $S_n :=$ set of all permutations on $[n] = \{1, \ldots, n\}$

So above $\Omega = S_{52}$; $|\Omega| = 52!$ (from Eg 6.1)

A = first 10 cards have a set of 4 aces.

Total # of aces = 4. Let $(\Omega, \mathbb{P})$ be unif. prob. space.

$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$ (UAR by assumption)

$A = \{(a_1, \ldots, a_{52}) \in \Omega : 1, 14, 27, 40 \in \{a_1, \ldots, a_{10}\}\}$

($\underset{\text{ace}}{1}, \ldots, 13$ - spade, $\underset{\text{ace}}{14} \cdots 26$ - clover, $\underset{\text{ace}}{27} \cdots 39$ - Hearts, $\underset{\text{ace}}{40} \cdots, 52$ - diamond)

$|A| = \binom{10}{4} 4! \, 48!$

$\binom{10}{4}$: # of positions for aces in first 10 slots; $4!$: ordering of aces; $48!$: remaining 48 cards however you like.

$\mathbb{P}(A) = \frac{10! \, 48!}{52! \, 6!} = \frac{10 \times 9 \times 8 \times 7}{52 \times 51 \times 50 \times 49} = \frac{9 \times 8}{52 \times 51 \times 35}$ ■

If we ask for prob. of 4 aces in first 20 positions?

$\mathbb{P}(A) = \binom{20}{4} / \binom{52}{4}$

AWAY FROM UNIFORM PROB. SPACES

Eg #2: Take simplest. $\Omega = \{0,1\}$.

A pmf p on Ω is $p(0), p(1) \geq 0$ & $p(0)+p(1) = 1$

Set $p(1) = p \in [0,1]$ (abuse of notation — $p: \Omega \to [0,1]$, $p \in [0,1]$)

Then $p(0) = 1-p$.

The PD $\mathbb{P}$ is called Bernoulli(p) prob. distribution.

Ex. A 2.7 Let (Ω_i, p_i), $i = 1, \ldots, k$ be finite prob. spaces.

$\Omega = \prod_{i=1}^{k} \Omega_i$ is the cartesian product.

Define $p((\omega_1, \ldots, \omega_k)) = \prod_{i=1}^{k} p_i(\omega_i) \quad \forall (\omega_1, \ldots, \omega_k) \in \Omega$.

Show that (Ω, p) is a finite prob. space

& $\mathbb{P}(A) = \prod_{i=1}^{k} \mathbb{P}_i(A_i) \quad \forall\ A = A_1 \times \cdots \times A_k \subseteq \Omega$.

$A = \{(\omega_1, \ldots, \omega_k) : \omega_i \in A_i \ \forall i\}$

"Product of prob. spaces with product pmf is a prob. space"

Eg 7.3 $\Omega = \{0,1\}^n$; Consider $(\{0,1\}, \mathbb{P}_i)$ to be Ber(p) prob. distribution $\forall\ i = 1, \ldots, n$

$p((\omega_1, \ldots, \omega_n)) = \prod_{i=1}^{n} p_i(\omega_i) = \prod_{i=1}^{n} p^{\omega_i}(1-p)^{1-\omega_i}$

$$\left(p_i(\omega_i) = p^{\omega_i}(1-p)^{1-\omega_i} = \begin{cases} p & \text{if } \omega_i = 1 \\ 1-p & \text{if } \omega_i = 0 \end{cases} \right) \qquad \left[\begin{matrix} (\Omega, p) \\ \parallel \\ (\Omega, \mathbb{P}) \end{matrix} \right]$$

$= p^{\sum_{i=1}^{n} \omega_i} (1-p)^{n - \sum_{i=1}^{n} \omega_i}$ ← [Product Ber(p) distribution on $\{0,1\}^n$]

[pmf on Ω depends only on $\sum_{i=1}^{n} \omega_i$]

If $p = 1/2$, this uniform PD on $\{0,1\}^n$ i.e., $p((\omega_1, \ldots, \omega_n)) = \frac{1}{2^n}$.

Eg 7.4 $\Omega = \{0, \ldots, n\}$. Define $\underset{\mathbb{P}(\{k\})}{\overset{\parallel}{\mathbb{P}(k)}} := \binom{n}{k} p^k (1-p)^{n-k}$, $p \in [0,1]$.

$\mathbb{P}$ is called the Binomial (n,p) distribution.

Check $\mathbb{P}(k) \geq 0$, $\sum_{k=0}^{n} \mathbb{P}(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$

$$= (p+1-p)^n = 1.$$

Eg 7.5: Let $\mathbb{P}$ be product Ber(p) distribution on $\{0,1\}^n$.

Define $f: \{0,1\}^n \longrightarrow \{0, \dots, n\}$ as $f(\omega_1, \dots, \omega_n) = \sum_{i=1}^{n} \omega_i$

$A_k = \{(\omega_1, \dots, \omega_n) : f(\omega_1, \dots, \omega_n) = k\}$ $\quad 0 \leq k \leq n$.
$\subseteq \{0,1\}^n$.

$$\mathbb{P}(A_k) = \sum_{\omega \in A_k} \mathbb{P}(\omega) \qquad \left(\mathbb{P}(\omega) = \mathbb{P}(\{\omega\}) = p(\omega)\right)$$

$$= \sum_{\omega \in A_k} p^{\sum \omega_i} (1-p)^{n - \sum \omega_i} \qquad \text{(by defn of } \mathbb{P} \text{ in Eg 7.3)}$$

$$= p^k (1-p)^{n-k} |A_k| \qquad (\textstyle\sum \omega_i = k \ \forall\, \omega \in A_k)$$

$$= \binom{n}{k} p^k (1-p)^{n-k} \qquad \left(|A_k| = \binom{n}{k} \text{ exactly } k \text{ '1's in } n \text{ slots}\right)$$

$\Omega = \bigsqcup_{k=0}^{n} A_k$ as $f(\omega) \in \{0, \dots, n\}$

$$1 = \mathbb{P}(\Omega) \underset{\text{(fin. add.)}}{=} \sum_{k=0}^{n} \mathbb{P}(A_k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}.$$

Eg 7.6 Consider $\Omega = \{0,1\}^n$ as above with product Ber (p) distribn.

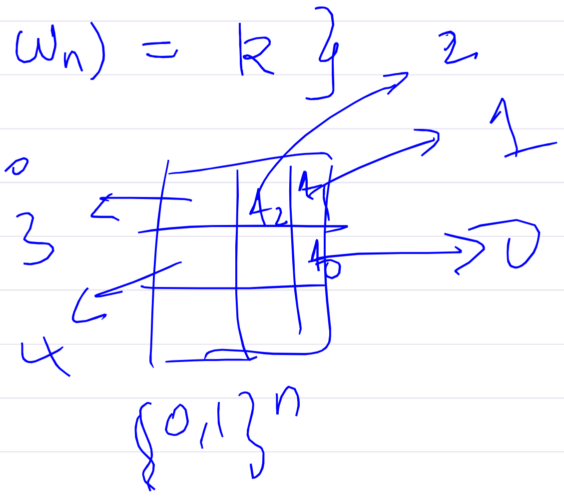
Let $f: \{0,1\}^n \longrightarrow \{0, \dots, n\}$ be a function.

$0 \leq k \leq n$. Define $A_k = \{(\omega_1, \dots, \omega_n) : f(\omega_1, \dots, \omega_n) = k\}$

A_k's are pairwise disjoint for $k = 0, \dots, n$.

& $\Omega = \bigsqcup_{k=0}^{n} A_k$

So $1 = \mathbb{P}(\Omega) = \sum_{k=0}^{n} \mathbb{P}(A_k)$ (fin. add.)



Suppose we define $\tilde{p} : \{0, \dots, n\} \Rightarrow [0,1]$
as $\tilde{p}(k) = \mathbb{P}(A_k)$.

Check $\tilde{p}$ is a pmf on $\{0, \dots, n\}$. Why? $\tilde{p}(k) \geq 0 \ \forall k$, $\sum_{k=0}^{n} \tilde{p}(k) = 1$.

So $(\{0, \dots, n\}, \tilde{p})$ is a finite prob. space.

$\Rightarrow \tilde{\mathbb{P}}(A) = \sum_{k \in A} \tilde{p}(k) \quad \forall A \subseteq \{0, \dots, n\}$.

So using $(\Omega, \mathbb{P})$ & $f : \Omega \to \{0, \dots, n\}$
we have constructed a new prob. space $(\{0, \dots, n\}, \tilde{p})$.
sample space

LEMMA 7.7 (Induced prob. lemma).
Let $(\Omega, \mathbb{P})$ be a finite prob. space & $\tilde{\Omega}$ is a finite set.
Let $f : \Omega \to \tilde{\Omega}$ be a function.

Define $\tilde{p}(\tilde{\omega}) := \mathbb{P}(\{\omega \in \Omega : f(\omega) = \tilde{\omega}\})$, $\tilde{\omega} \in \tilde{\Omega}$.
$= \mathbb{P}(f^{-1}(\tilde{\omega}))$
$[f^{-1}(\tilde{\omega}) = \{\omega \in \Omega : f(\omega) = \tilde{\omega}\}]$

Then we have that $(\tilde{\Omega}, \tilde{p})$ is a fin. prob. space.

Proof: To show $(\tilde{\Omega}, \tilde{p})$ is a fin prob. space
we have to show $\tilde{p}$ is a pmf on $\tilde{\Omega}$.
i.e., $\tilde{p}(\tilde{\omega}) \geq 0$, $\sum_{\tilde{\omega} \in \tilde{\Omega}} \tilde{p}(\tilde{\omega}) = 1$.

EXTRA: Given $a_0, \dots, a_n \geq 0$. To check $\sum_{i=0}^{n} a_i = 1$
Construct $(\Omega, \mathbb{P})$ & events $A_0, \dots, A_k$
$\ni \Omega = \bigsqcup_{k=0}^{n} A_k$ & $a_k = \mathbb{P}(A_k) \ \forall k = 0, \dots, n$.

Qn: Given $a_0, \dots, a_n \geq 0$ & $\sum_{i=0}^{n} a_i = 1$.

$\Omega = \{0,1\}^n$ & $\mathbb{P}$ – product Ber(p) distribution.

– ~~then~~ Can we always find p & $A_0, \dots, A_n$ ∋

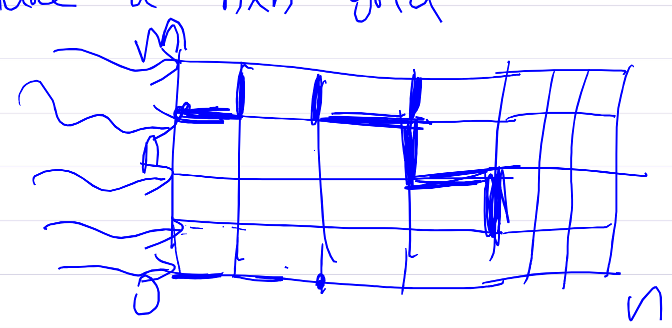
$$a_k = \mathbb{P}(A_k) \,?$$

Try: $p = 1/2 \Rightarrow \mathbb{P}$ – Uniform PD on Ω.

$$\mathbb{P}(A_k) = \frac{|A_k|}{2^n}.$$

Can I find A_k's ∋ $a_k = \frac{|A_k|}{2^n}$? True if $2^n a_k \in \mathbb{N} \cup \{0\}$.

[OPEN PROBLEM] Take a $n \times n$ grid [Percolation theory]



$$= \left\{ (i,j) : \begin{array}{l} 0 \leq i \leq n \\ 0 \leq j \leq m \end{array} \right\}$$

E = set of pipes $\overset{n}{=} \{ ((i,j) \to (i,j+1)), ((i,j) \to (i+1,j)) : \forall\, i,j \}$

$\Omega = 2^E$ = Collection of subset of pipes.

$\mathbb{P}(\omega) = \frac{1}{2^{|E|}}$, $\omega \subseteq E$ $\quad N = |E| = n(m+1) + m(n+1)$

prob. ω is not destroyed.

$\mathbb{P}$(Water flows from Left to right) = ?? $\quad m = 2n, 3n$?

S. Smirnov solved it in Triangular lattice