26/10 LECTURE 8.

Recall $(\Omega, p)$ or $(\Omega, \mathbb{P})$ is a finite prob. space.
$p$: pmf, $\mathbb{P}$: PD

obvious example $p(\omega) = \frac{1}{|\Omega|}$ $\omega \in \Omega$ (Eq. likely outcomes / UAR).

Job: Find $A \subseteq \Omega$ & compute $\mathbb{P}(A)$.

Eg. $\mathrm{Ber}(p)$ $p(0) = 1 - p = 1 - p(1)$ $\Omega = \{0, 1\}$. [coin toss head prob. $p$]

Eg. $\mathrm{Bin}(n, p)$ $\Omega = \{0, \ldots, n\}$ $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$ [# of heads in $n$ biased coin tosses]

Eg 8.1: (A3) $\mathrm{Hyg}(n, m, r)$ (Hypergeometric) $\Omega = [n] \cup \{0\}$; $r \leq m \leq n$.

$k \in \Omega$.

$$p(k) = \begin{cases} \dfrac{\binom{m}{k}\binom{n-m}{r-k}}{\binom{n}{r}} & k \leq r \text{ \& } k \leq m \\ 0 & \text{else.} \end{cases}$$

Context: Choose $r$ objects out of $n$. There are exactly $m$ 'good' ones

[see computations in sampling without replacement]

Prob. (one has chosen $k$ good objects) $= p(k)$

Show $(\Omega, p)$ is a prob. space.

To show $(\Omega, p)$ (in general) is a prob. space

- Show $p$ is a pmf
- show $\mathbb{P}$ is a PD.
- Use induced prob. lemma.
- use "product" of prob. spaces [A 2.7]
- use "projection" $(\omega_1, \omega_2) \mapsto \omega_1$ [A 2.8]

$[\Omega_1 \times \Omega_2 \longrightarrow \Omega_1]$

projection map

[check: std. case of Ind. prob. lemma]

For eg. $Bin(n, p)$ — Verify $Ber(p)$ is a PD/pmf induced prob. lemma: construct $\Downarrow$ product space

Also directly check $\mathrm{Bin}(n,p)$ is a PD.

Eg 8.2 (Maxwell-Boltzmann distribution)
(A3)

$$\Omega_{r,n} = \left\{ (r_1, \ldots, r_n) : r_i \geq 0 \quad \sum_{i=1}^{n} r_i = r \right\}$$

r unlabelled particles into n labelled cells.

Define $p(\omega) := \dfrac{r!}{r_1! \cdots r_n!} \dfrac{1}{n^r}$ $\quad \omega = (r_1, \ldots, r_n)$

show $(\Omega_{r,n}, p)$ is a PS.

COMPARISON: BE $\quad p(\omega) = \dfrac{1}{|\Omega_{r,n}|} \quad \omega \in \Omega_{r,n}$.

FD $\quad p(\omega) = \dfrac{1}{|\Omega^*_{r,n}|} \quad \omega \in \Omega^*_{r,n}$ (0 else)

---

If you solve A3, you will understand why MB came first & very natural mathematically.

Photons "follow" BE & Electrons "follow" FD.

Eg 8.3 [Boltzmann-Gibbs distribution] $\Omega$ - finite set.

Let $H : \Omega \to \mathbb{R}$ be a function. $\beta \geq 0$

Define $Z_\beta := \sum_{\omega \in \Omega} e^{-\beta H(\omega)}$

Define $p(\omega) := \dfrac{e^{-\beta H(\omega)}}{Z_\beta}$ $\omega \in \Omega$.

Check $(\Omega, p)$ is a PS. $\left( p(\omega) \geq 0 \ \& \ \sum_{\omega \in \Omega} p(\omega) = 1 \right)$

Interpretation: $\omega$ - a state

$H(\omega)$ - energy of the state. $\beta = \frac{1}{kT}$ T - temp $T \geq 0$

low temp $T = 0 \iff \beta = \infty \to$ system UAR on minimal energy state

high temp $T = \infty \iff \beta = 0 \to$ system choses a state UAR.

If $\beta = 0$ then $Z_\beta = |\Omega|$ & $p(\omega) = \frac{1}{|\Omega|}$ (UAR)

Show (Analysis ex:)

$$\lim_{\beta \to \infty} p(\omega) = \frac{\mathbb{1}[\omega \in \Omega_*]}{|\Omega_*|}$$ where $\Omega_* := \{\omega \in \Omega : H(\omega) = \min_{\delta \in \Omega} H(\delta)\}$

$$\left( \mathbb{1}[\omega \in \Omega_*] = \begin{cases} 1 & \text{if } \omega \in \Omega_* \\ 0 & \text{else} \end{cases} \right)$$

Indicator function

At any temperature $\beta \in (0, \infty)$, system likes to be in minimal energy states but may not be.

Penalise a non min. energy state at small temperatures (ie. $T$ close to $0$)

---

Ex. 8.4 Let $(\Omega, P)$ be a PS $\ni$ $p(\omega) > 0$ $\forall$ $\omega \in \Omega$.

Show that $\exists$ $H : \Omega \to \mathbb{R}$ & $\beta \geq 0$ $\ni$ $p(\omega) = \dfrac{e^{-\beta H(\omega)}}{Z_\beta}$.

[ Giorgio Parisi Nobel (2021) in Physics for spin systems.

$\Omega = \{-1, +1\}^n$ $H : \Omega \to \mathbb{R}$ is random ]

Qn: How to compute $\mathbb{P}(A)$ given $(\Omega, \mathbb{P})$?

- we know some examples.
- what are our tools?

Recall

THM 4.1 Let $(\Omega, \mathbb{P})$ be a fin. prob. space. Then the following hold.

(i) $\mathbb{P}(A) \le \mathbb{P}(B) \quad \forall \; A \subseteq B \subseteq \Omega$. [Monotonicity]

(ii) $\mathbb{P}(A) \le 1 \quad \forall \; A \subseteq \Omega$

(iii) $\mathbb{P}(\phi) = 0$

(iv) $\mathbb{P}(A_1 \cup \cdots \cup A_n) \le \sum_{i=1}^{n} \mathbb{P}(A_i) \quad \forall \; A_1, \ldots, A_n \subseteq \Omega$. [finite subadditivity]

(v) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ [Inclusion-Exclusion]

+ finite additivity.

**THM 8.5:** Let $(\Omega, \mathbb{P})$ be prob. space. Let $A_1, \ldots, A_n$ be events.

[I-E formula]

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = S_1 - S_2 + S_3 - \cdots + (-1)^{n-1} S_n$$

$$S_k := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$$

**Proof:** $S_1 = \sum_{i=1}^{n} \mathbb{P}(A_i)$ $\quad S_2 = \sum_{1 \le i < j \le n} \mathbb{P}(A_i \cap A_j)$.

$S_n = \mathbb{P}(A_1 \cap \cdots \cap A_n)$

We'll prove by induction.

I-E True for $n = 1, 2$.

Assume I-E holds upto $(n-1)$ for $n \ge 3$.

We'll prove for $n$.

Write $B = A_2 \cup \cdots \cup A_n$.

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \mathbb{P}(A_1 \cup B) = \mathbb{P}(A_1) + \mathbb{P}(B) - \mathbb{P}(A_1 \cap B) \quad -(1)$$

$$\mathbb{P}(B) = \mathbb{P}(A_2 \cup \cdots \cup A_n)$$

$$= S_1' - S_2' + S_3' \cdots + (-1)^{n-2} S_{n-1}' \quad (\text{I-E for } n-1) \quad -(2)$$

$$S_k' = \sum_{2 \le i_1 < \cdots < i_k \le n} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$$

$$A_1 \cap B = \bigcup_{i=2}^{n} (A_1 \cap A_i)$$

Use I-E for $n-1$ on $A_1 \cap B$

$$\mathbb{P}(A_1 \cap B) = \tilde{S}_1 - \tilde{S}_2 + \cdots + (-1)^{n-2} \tilde{S}_{n-1} \quad -(3)$$

$$\tilde{S}_k = \sum_{2 \le i_1 < \cdots < i_k \le n} \mathbb{P}(A_1 \cap A_{i_1} \cap \cdots \cap A_{i_k})$$

From (1), (2) & (3)

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \mathbb{P}(A_1) + S_1' - S_2' + S_3' \cdots + (-1)^{n-2} S_{n-1}' + 0$$

$$0 - \tilde{S}_1 + \tilde{S}_2 \cdots + (-1)^{n-1} \tilde{S}_{n-1}$$

Check

$$\mathbb{P}(A_1) + S_1 = \mathbb{P}(A_1) + \sum_{i=2} \mathbb{P}(A_i) = S_1$$

$$-S_2' - \tilde{S}_1 = -S_2$$

$$\cdots\cdots$$

$$(-1)^{n-2} S'_{n-1} + (-1)^{n-2} \tilde{S}_{n-2} = S_{n-1} (-1)^{n-2}$$

$$(-1)^{n-1} \tilde{S}_{n-1} = (-1)^{n-1} S_n \qquad [\tilde{S}_{n-1} = S_n].$$

$$S_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$$

$$= \sum_{2 \le i_2 < i_3 < \cdots < i_k \le n} \mathbb{P}(A_1 \cap A_{i_2} \cap \cdots \cap A_{i_k}) + \sum_{2 \le i_1 < i_2 < \cdots < i_k < n} \mathbb{P}(A_{i_1} \cap \cdots A_{i_k})$$

(Fix $i_1 = 1$) (Take $i_1 > 1$)

(by defns) $= \tilde{S}_{k-1} + S'_k$

$$(-1)^{k-1} S_k = (-1)^{k-1} S'_k - (-1)^{k-2} \tilde{S}_{k-1}$$

$\boxed{}$

Eg: $n=3$

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B \cup C) - \mathbb{P}(A \cap (B \cup C))$$ (IF for $n=2$)

$$= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) - \mathbb{P}((A \cap B) \cup (A \cap C))$$

$$= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) + \mathbb{P}(A \cap B \cap C).$$

Ex. (Bonferroni Inequalities). Assume as in I-E.

Then $$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{k=1}^{m} (-1)^{k+1} S_k \quad \forall \text{ odd } m.$$

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \ge \sum_{k=1}^{m} (-1)^{k+1} S_k \quad \forall \text{ even } m.$$