28/10 LECTURE 9: Applications of I-E & Bonferroni

THM 8.5: Let $(\Omega, \mathbb{P})$ be prob. space. Let $A_1, \dots, A_n$ be events.

[I-E formula]

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = S_1 - S_2 + S_3 - \dots + (-1)^{n+1} S_n$$

$$S_k := \sum_{1 \le i_1 < i_2 < \dots < i_k \le n} \mathbb{P}(A_{i_1} \cap \dots \cap A_{i_k})$$

Eg 9.1: r labelled balls are thrown randomly into n labelled cells.

$$P_m(r,n) = \mathbb{P}(\text{exactly } m \text{ cells are empty}), \quad m = 0, \dots, n-1$$

$$\Omega = \{(\omega_1, \dots, \omega_r) : \omega_i \in [n]\}, \quad |\Omega| = n^r$$

ω_1 ↳ cell of first ball.

$$p(\omega) = \frac{1}{n^r} \quad (\text{eq. likely})$$

Take $m = 0$: $A_j = j^{th}$ cell is empty $= \{\omega : \omega_i \neq j \ \forall i = 1, \dots, r\}$

no cell is empty $= \bigcap_{j=1}^{n} A_j^c = \left(\bigcup_{j=1}^{n} A_j\right)^c$

$$p_0(r,n) = \mathbb{P}(\text{no cell is empty}) = \mathbb{P}\left(\bigcap_{j=1}^{n} A_j^c\right)$$

$$= \mathbb{P}\left(\left(\bigcup_{j=1}^{n} A_j\right)^c\right) = 1 - \mathbb{P}\left(\bigcup_{j=1}^{n} A_j\right) \quad \left(\mathbb{P}(A) + \mathbb{P}(A^c) = 1\right) \quad — (1)$$

To compute $\mathbb{P}\left(\bigcup_{j=1}^{n} A_j\right)$ we'll use I-E.

$$\mathbb{P}(A_1 \cap \cdots \cap A_k) = \frac{|A_1 \cap \cdots \cap A_k|}{n^r} \quad (\text{UAR})$$

$$|A_1 \cap \cdots \cap A_k| = |\{\omega : \omega_i \neq 1, \ldots, k \quad \forall i = 1, \ldots, r\}|$$

$$= (n-k)^r$$

$$\mathbb{P}(A_1 \cap \cdots \cap A_k) = \left(1 - \frac{k}{n}\right)^r$$

Same way $\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \left(1 - \frac{k}{n}\right)^r \quad 1 \le i_1 < \cdots < i_k \le n.$

by defn of S_k, $S_k = \binom{n}{k}\left(1 - \frac{k}{n}\right)^r$ — (2)

$$P(A_1 \cup \dots \cup A_n) \underset{(IE)}{=} \sum_{k=1}^{n} (-1)^{k-1} S_k$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \left(1 - \frac{k}{n}\right)^r \qquad (\text{using } 2)$$

$$p_0(r,n) = 1 - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \left(1 - \frac{k}{n}\right)^r \qquad (\text{using } 1)$$

$$|\{\omega : \text{no cell is empty}\}| = n^r p_0(r,n) \qquad (\text{by defn of } p_0 \text{ \& UAR})$$

$\underline{m \geq 1}$

$$p_m(r,n) = P(\text{exactly } m \text{ cells are empty})$$

Exactly m cells are empty

→ choose exactly m cells to be empty. ← $\binom{n}{m}$ ways.

→ r balls are to be thrown into remaining $n-m$ cells such that none of them are empty. ← $(n-m)^r p_0(r, n-m)$ ways.

$$p_m(r,n) = \frac{\binom{n}{m}(n-m)^r \, p_0(r, n-m)}{n^r}$$

$$= \binom{n}{m} \sum_{k=0}^{n-m} (-1)^k \frac{(n-m)^r}{n^r} \binom{n-m}{k} \left(1 - \frac{k}{n-m}\right)^r$$ (substitute $p_0(r,n-m)$ value)

$$= \binom{n}{m} \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \left(1 - \frac{(m+k)}{n}\right)^r$$

Ex* Fix m

Q. 2 Find $\lim_{n\to\infty} p_m(r,n) = ?$

Case 1: Fix r : Case 2: let $r \to \infty$ $\ni$ $\frac{r}{n} \to 0$

Case 3: let $r \to \infty$ & $ne^{-r/n} = \lambda \in (0,\infty)$ fixed.

$n = 365$. $p_m(r,n) = P($Exactly m days have no birthdays among r people$)$

THM 9.3 (Bonferroni's inequalities) Let $A_1, \ldots, A_n$ be events in a prob. space $(\Omega, \mathbb{P})$ & $A = \bigcup_{i=1}^{n} A_i$. Then

$$\mathbb{P}(A) \le \sum_{k=1}^{m} (-1)^{k-1} S_k \quad \text{for odd } m$$

$$\mathbb{P}(A) \ge \sum_{k=1}^{m} (-1)^{k-1} S_k \quad \text{for even } m.$$

Eg 9.4 (Contd. of Eg 9.2) Find bounds $\mathbb{P}$(at least one cell is empty)

A_j = j^{th} cell is empty.

$A = \bigcup_{j=1}^{n} A_j$ = at least one cell is empty.

From Bonferroni's ineq. $\quad S_1 - S_2 \le \mathbb{P}(A) \le S_1$

$$S_1 = n\left(1-\frac{1}{n}\right)^r \qquad S_2 = \binom{n}{2}\left(1-\frac{2}{n}\right)^r$$

$n = 10$; $r = 40$; $\quad 0.1418 \le p_1(r, n) \le 0.1478$

Ex* Find out for what range of r, n $S_1 - S_2 \approx S_1$ are close

Proof of THM 9.3: (sketch).

We'll prove for $m=1,2$ & rest is exercise.

For $m=1$ $\mathbb{P}(A) \le S_1$ (finite subadditivity / Union bound)

$m=2$: $\mathbb{P}(A) = \sum_{\omega \in A} p(\omega)$ (defn)

$= \sum_{\omega \in \Omega} p(\omega) \mathbb{1}_A(\omega)$ $\left[\begin{array}{l} \mathbb{1}_A : \Omega \to \{0,1\} \\ \omega \mapsto \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{else} \end{cases} \\ \text{indicator} \end{array} \right]$

Monotonicity: Suppose we show that $\mathbb{1}_A(\omega) \ge f(\omega) \ \forall \omega \in \Omega$

for some function $f : \Omega \longrightarrow \mathbb{R}$

Then since $p(\omega) \ge 0 \ \forall \omega \in \Omega$, we have $\sum_{\omega \in \Omega} p(\omega) \mathbb{1}_A(\omega) \ge \sum_{\omega \in \Omega} f(\omega) p(\omega)$

Assume $\mathbb{1}_A(\omega) \geq \sum_{i=1}^{n} \mathbb{1}_{A_i}(\omega) - \sum_{1\le i_1 < i_2 \le n} \underbrace{\mathbb{1}_{A_{i_1}}(\omega)\, \mathbb{1}_{A_{i_2}}(\omega)}_{= f(\omega)}$ — (4)

Note that $\mathbb{1}_{A_{i_1}}(\omega)\,\mathbb{1}_{A_{i_2}}(\omega) = \mathbb{1}_{A_{i_1}\cap A_{i_2}}(\omega)$

If (4) holds, from monotonicity we have that

$$\mathbb{P}(A) = \sum_{\omega\in\Omega} p(\omega)\,\mathbb{1}_A(\omega)$$

$$\geq \sum_{\omega\in\Omega} p(\omega)\left(\sum_{i=1}^{n} \mathbb{1}_{A_i}(\omega) - \sum_{1\le i_1<i_2\le n} \mathbb{1}_{A_{i_1}\cap A_{i_2}}(\omega)\right)$$

$$= \sum_{\omega\in\Omega} p(\omega) \sum_{i=1}^{n} \mathbb{1}_{A_i}(\omega) - \sum_{\omega\in\Omega} p(\omega) \sum_{1\le i_1<i_2\le n} \mathbb{1}_{A_{i_1}\cap A_{i_2}}(\omega)$$

(interchange sums) $$= \sum_{i=1}^{n}\sum_{\omega\in\Omega} \mathbb{1}_{A_i}(\omega)\,p(\omega) - \sum_{1\le i_1<i_2\le n}\sum_{\omega\in\Omega} p(\omega)\,\mathbb{1}_{A_{i_1}\cap A_{i_2}}(\omega)$$

(justified as we are dealing with finitely many terms)

$$= \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{1 \le i_1 < i_2 \le n} \mathbb{P}(A_{i_1} \cap A_{i_2})$$

$$= S_1 - S_2.$$

Proof for $m=2$ is complete assuming (4)

Let us prove (4).

Case 1: $\omega \notin A$. $\mathbb{1}_A(\omega) = 0$; $\mathbb{1}_{A_i}(\omega) = 0$ as $A_i \subseteq A$.

Case 2: $\omega \in A$. Assume ω is in exactly r many A_i's for some $r \ge 1$.

Without loss of generality (WLOG), assume $\omega \in A_1 \cap \cdots \cap A_r$ & $\omega \notin A_i \ \forall i > r$.

$\mathbb{1}_A(\omega) = 1$; $\sum_{i=1}^{n} \mathbb{1}_{A_i}(\omega) = r$, $\sum_{1 \le i_1 < i_2 \le n} \mathbb{1}_{A_{i_1} \cap A_{i_2}}(\omega) = \binom{r}{2}$

So $f(\omega) = r - \binom{r}{2} = r\left(1 - \frac{r-1}{2}\right) \le 1$ if $r \ge 1$

So $\quad 1_A(\omega) \geq f(\omega) \quad \forall\, \omega \in A.$

Case 1 + Case 2 $\implies$ (4) is true.

So proof is complete for $m = 2$.

General m — Ex. (A4). ∎

Ex. (A4) Prove I-E using above approach.

See Feller (Ch 5) & Venkatesh (Ch IV) for more applications.

Ex: (A4) $\mathbb{P}(\text{exactly } m \text{ of } A_1, \dots, A_n \text{ occur}) = \sum_{k=0}^{n-m} (-1)^k \binom{m+k}{m} S_{m+k}$

Ex: A4 $\mathbb{P}(\text{at least } m \text{ of } A_1, \dots, A_n \text{ occur}) = \dots$