3/12/21

Agenda: · Assignment 6 solns. with motivation.
· Some selected problems from Ross, Chapter 4.

(Q2) Prove the following simpler assertion first:
"If $A_1, \dots, A_n$ are ind., then $A_2 \cup A_1, A_3, \dots, A_n$ are ind."

Pf: If $S \subseteq [n-1]$, & S doesn't contain the index 1, then it is clear.
Otherwise let $S = \{1\} \uplus J$, $J \subseteq \{2,3,\dots,n-1\}$. Note that:

$$\mathbb{P}((A_2 \cup A_1) \cap (\bigcap_{j\in J} A_j)) = \mathbb{P}(\bigcap_{j\in J} A_j) - \mathbb{P}((A_2 \cup A_1)^c \cap (\bigcap_{j\in J} A_j))$$

$$= \mathbb{P}(\bigcap_{j\in J} A_j) - \mathbb{P}(A_2^c \cap A_1^c \cap (\bigcap_{j\in J} A_j))$$

$$= \prod_{j\in J} \mathbb{P}(A_j) - \left[\prod_{j\in J} \mathbb{P}(A_j)\right]\left[\mathbb{P}(A_1^c)\mathbb{P}(A_2^c)\right]$$

$$= \prod_{j\in J} \mathbb{P}(A_j)\left[1 - \mathbb{P}(A_1^c \cap A_2^c)\right] = \left[\prod_{j\in J} \mathbb{P}(A_j)\right]\left[\mathbb{P}(A_1 \cup A_2)\right].$$

Thus, we are done. Now, inductively repeat this prop'n to finish.

(6)
(i) If the k people are fixed, then we look at each possible edge being present i.e. $p^{\binom{k}{2}}$. Let S be the set.
(ii) For each person not in S, the probability they don't connect with at least one person in S is $(1-p^k)$, so we get $p^{\binom{k}{2}}(1-p^k)^{n-k}$.

(7) $u_1 = \mathbb{P}(AA) = \mathbb{P}(\text{father } AA, \text{ mother } AA)$ u^2
$+ \frac{1}{2}[\mathbb{P}(\text{father } AA, \text{ mother } Aa) + \mathbb{P}(\text{father } Aa, \text{ mother } AA)]$ $4uv$
$+ \frac{1}{4}[\mathbb{P}(\text{father } Aa, \text{ mother } Aa)]$ v^2
$= u^2 + v^2 + 2uv$

$w_1 = \mathbb{P}(aa) = w^2 + v^2 + 2vw$

$2v_1 = 1 - u_1 - w_1 = (u+2v+w)^2 - u^2 - w^2 - 2v^2 - 2uv - 2wv$
$= 2v^2 + 2uv + 2wv + 2uw$

(9) r red, b black balls, sampling with addition of same ball. Suppose b_n, r_n black/red balls are in the urn before n^{th} draw.
Let $B_k = \{k^{th} \text{ ball is black}\}$.
Then $\mathbb{P}(B_n \cap B_m) = \mathbb{P}(B_n | B_m)\mathbb{P}(B_m)$.
$\mathbb{P}(B_m)$ is constant in m: $\mathbb{P}(B_{m+1}) = \mathbb{P}(B_{m+1}|B_m)\mathbb{P}(B_m) + \mathbb{P}(B_{m+1}|B_m^c)\mathbb{P}(B_m^c)$

$$= \frac{(b_m+2)(b_m+1)}{(b_m+r_m+2)(b_m+r_m+1)} + \frac{(b_m+1)r_m}{(b_m+r_m+2)(b_m+r_m+1)} = \frac{(b_m+1)}{(b_m+r_m+1)} = \mathbb{P}(B_m),$$

$\Rightarrow$ by induction, $\mathbb{P}(B_m) = \mathbb{P}(B_0) = b/b+r$.
We claim that $\mathbb{P}(B_n|B_m) = b+c/b+r+c \quad \forall n > m$.
To prove this: take $m=1$, then $\mathbb{P}(B_n|B_1) = \mathbb{P}'(B_{n-1})$ where $\mathbb{P}'$ is the prob. for the Polya urn experiment starting with $b+r+c$ total no. of balls & $b+c$ black balls. So, $\mathbb{P}'(B_{n-1}) = b+c/b+r+c$. Now we assume that $\mathbb{P}(B_d|B_f) = b+c/b+r+c \;\; \forall d > f, f \in \{1,2,\dots,k\}$. We will now prove this for $k+1$. So:

$\rightarrow \mathbb{P}(B_d|B_{k+1}) = \mathbb{P}(B_d \cap B_1|B_{k+1}) + \mathbb{P}(B_d \cap B_1^c|B_{k+1})$

$$= \mathbb{P}(B_d|B_{k+1} \cap B_1)\mathbb{P}(B_{k+1}|B_1)\frac{\mathbb{P}(B_1)}{\mathbb{P}(B_{k+1})} + \mathbb{P}(B_d|B_{k+1} \cap B_1^c)\mathbb{P}(B_{k+1}|B_1^c)\frac{\mathbb{P}(B_1^c)}{\mathbb{P}(B_{k+1})}.$$

$= \mathbb{P}'(B_{d-1}|B_k) = b+2c/b+r+2c$
$= \mathbb{P}'(B_k) = \frac{b+c}{b+r+c}$
1
$\mathbb{P}''(B_{d-1}|B_k) = b+c/b+r+2c$
$\mathbb{P}''(B_k) = b/b+r+c$
$\frac{r/b+r}{b/b+r}$

$$= \frac{(b+2c)(b+c)}{(b+r+c)(b+r+2c)} + \frac{b(b+c)r}{b(b+r+c)(b+r+2c)} = (b+c)/(b+r+c).$$

Completing the induction. Thus, $\mathbb{P}(B_n \cap B_m) = b/b+r \times b+c/b+r+c$.

Properties of Binomial [Ross]

Pr. 6.1: The Binomial distr. is unimodal at $k/2$ for k even, and bimodal at $k\pm1/2$ for k odd.

Key takeaway: A Bin$(k, 1/2)$ distributed random variable is very likely to concentrate around $k/2$ i.e. events of the form $\{\text{Bin}(k, 1/2) \in [k/2 \pm \varepsilon]\}$ are likely to have large prob.

Can we prove this using the Pascal triangle interpretation or otherwise? A combinatorially proved inequality is an inequality between $A < B$, proved by finding two sets S_1, S_2 with $|S_1| = A$, $|S_2| = B$ & there is an $f: S_1 \to S_2$ injective.
Simple: Prove $\binom{n}{2} < \binom{n}{3} \;\forall n > 6$ using a combinatorial idea.

6(g) Is meant to demonstrate the definition of a Binomial RV: key takeaway is that $(1-p), p$ are often factored out of inequalities with binomial random variables.
Key qn: Is it possible to avoid computation?
Answer: Yes, using the concept of stochastic dominance. I'll send a link to this.

6(h) It is not surprising, with the description of the model, that your vote is more influential if you live in a larger state. What is surprising is the $\sqrt{n}$ dependence: even if you are capable of influencing m votes, your vote is likely to be decisive only $\sqrt{n}$ of the time. $\sqrt{n}$ is a recurring theme, particularly because of Stirling's approximation. More intuition will be provided in the next course.