7/1/22

Agenda: Exam answers, and intro to prob 2.

Answers [Brief]

Part (A)

(1) If X, Y are the results upon rolling a dice twice, then X, Y obviously have the same distribution i.e. $\text{Unif}[6]$, whose expectation is $\frac{6+1}{2}$. So $\mathbb{E}[X] = \mathbb{E}[Y] = 7/2$.

(2) The only conditions that are required to preserve pmfs are non-negativity and $\sum_{x\in\Omega} f(x) = 1$. So, squaring is unlikely to preserve pmfs. On the other hand, $\frac{p+q}{2}$ is a pmf, and $\sum_{x\in\Omega} p/2\,(x) = 1/2$ So it can never be a pmf.

(3) If $A \subseteq B$, then $A\cap B = A$, $A \cup B = B$. Therefore, $A \cap B^c = \phi$. We easily see that $\mathbb{P}(B|A) = \mathbb{P}(A^c|B^c) = 1$ and $\mathbb{P}(A|B^c) = 0 = \mathbb{P}(B^c|A)$. That $\mathbb{P}(A|B) = 1 - \mathbb{P}(A^c|B)$ is a swift consequence of the finite additivity of $\mathbb{P}(\cdot|B)$.

(4) It's about the quantities $\lim_{x\to-\infty} F(x)$, $\lim_{x\to\infty} F(x)$ because the candidates are all continuous. But:
$\lim_{x\to-\infty} e^{-x} = \infty$, $\lim_{x\to-\infty} 1 - \frac{e^{-x}}{2} = -\infty$, $\lim_{x\to-\infty} 1-2e^{-|x|} = 1$.

and $\lim_{x\to\infty} 1/_{1+x^2} = 0$: So <u>NONE</u> of them are DFs.

(5) According to Defn 25.7 and Thm. 25.10, two PDFs represent the same random variable even if they differ at one point. Now, the operations $x \to x+1$ and $x \to x^2$ are continuous: so $Y = X+1$, $Z = X^2$ are also continuous rvs. Having said that, it is not necessary that Z needs to have a PDF with $f_Z(z) = 0\ \forall z \le 0$, even if $\mathbb{P}(Z \le 0) = 0$. For example, take the two PDFs:

$f_X = \mathbb{1}_{\{0<x<1\}}$ $\qquad f'_X = \mathbb{1}_{\{0<x<1\}} + \mathbb{1}_{\{x=2\}}$

They're both pdfs: look at 25.7. But both are a pdf for the uniform $[0,1]$ random variable!? So, $f_Z(z) = 0\ \forall z \le 0$ is false. $f_Y(y) = 0\ \forall y \le 1$ is false anyway, let $X \sim U[-1,0]$ e.g.

Part B

(1) The key point: $A_{k_1} \cap \cdots \cap A_{k_m} = A_{\text{lcm}(k_1,\ldots,k_m)}$. Indeed, j is a multiple of $k_1 \cdots k_m$ if and only if it is a multiple of $\text{lcm}(k_1 \cdots k_m)$. Thus, combining with $\mathbb{P}(A_i) = 1/i$ $\forall i$ dividing n

$$\mathbb{P}(A_{k_1} \cap \cdots \cap A_{k_m}) = \mathbb{P}(A_{\text{lcm}(k_1\cdots k_m)}) = \frac{1}{\text{lcm}(k_1 \cdots k_m)}$$

We now see that if $\{k_1, k_2, k_3\}$ are pairwise coprime, then $\forall S \subseteq \{1,2,3\}$ we have

$$\text{lcm}(\{k_s : s \in S\}) = \prod_{s\in S} k_s$$

and therefore, as a consequence, $\mathbb{P}(\bigcap_{s\in S} A_s) = \prod_{s\in S} \mathbb{P}(A_s)$, so $\{A_{k_i} : i = 1,2,3\}$ is an independent colln. However, if k_1, k_2 are not coprime, then $A_{k_1} \cap A_{k_2} \ne A_{k_1 k_2}$ so clearly A_{k_1} is <u>not independent</u> of A_{k_2}.

2) An exercise in cond'n'l probability. Define the events:

$E_i = \{i^{th} \text{ ball is red}\}$

$E_{ij} = \{i^{th}\ \&\ j^{th} \text{ balls are red}\}$, $i<j$

$E_{ijk} = \{i^{th}, j^{th}\ \&\ k^{th} \text{ balls are red}\}$, $i<j<k$.

Then $X_{ijk} = \mathbb{1}_{E_{ijk}}$, so $\mathbb{E}[X_{ijk}] = \mathbb{P}(E_{ijk})$ and $\text{Var}(X_{ijk}) = \mathbb{P}(E_{ijk})\mathbb{P}(E_{ijk}^c)$.

(1) $\mathbb{P}(E_1) = r/b+r$. Claim: $\mathbb{P}(E_i) = r/b+r\ \forall i$.

Pf: $\mathbb{P}(E_i) = \underbrace{\mathbb{P}(E_i|E_1)}_{r+c/b+r+c}\underbrace{\mathbb{P}(E_1)}_{r/b+r} + \underbrace{\mathbb{P}(E_i|E_1^c)}_{r/b+r+c}\underbrace{\mathbb{P}(E_1^c)}_{b/b+r}$

$E_{i-1}^{(r+c,b)}$ $\qquad E_{i-1}^{(r,b+c)}$

$= r/b+r$.

(2) $\forall i > 1$, $\mathbb{P}(E_{1i}) = \mathbb{P}(E_i|E_1)\mathbb{P}(E_1) = \frac{r(r+c)}{(b+r)(b+r+c)}$

<u>Claim:</u> $\mathbb{P}(E_{ij}) = r(r+c)/(b+r)(b+r+c)$ $\forall i<j$.

Pf: $\mathbb{P}(E_{ij}) = \mathbb{P}(E_j|E_i)\mathbb{P}(E_i)$.

$\mathbb{P}(E_j|E_i) = \mathbb{P}(E_j \cap E_1|E_i) + \mathbb{P}(E_j \cap E_1^c|E_i)$

$= \mathbb{P}(E_{ij}|E_1)\mathbb{P}(E_1|E_i) + \mathbb{P}(E_{ij}|E_1^c)\mathbb{P}(E_1^c|E_i)$

Use induction hypothesis along with reasoning of (1) to conclude.

(3) For this, see that

$\mathbb{P}(E_{1jk}) = r(r+c)(r+2c)/(b+r)(b+r+c)(b+r+2c)$

Then use:

$\mathbb{P}(E_{jk}|E_l)$

$= \mathbb{P}(E_{ljk}|E_1)\mathbb{P}(E_1|E_l)$

$\quad + \mathbb{P}(E_{ljk}|E_1^c)\mathbb{P}(E_1^c|E_l)$

along with ind. hyp. & reasoning of parts (1),(2) to conclude that

$\mathbb{P}(E_{ijk}) = r(r+c)(r+2c)/(b+r)(b+r+c)(b+r+2c)$

(3) An exercise in binomials.

(a) $\{\text{System working}\} = \{\exists S \subseteq [5], |S| \ge 3$, exactly the components with labels in S are working$\}$

$\Rightarrow \mathbb{P}(\{\text{System working}\}) = \sum_{\substack{S\subseteq[5]\\|S|\ge 3}} \prod_{i\in[5]} (\mathbb{1}_{i\in S}\, p_i + \mathbb{1}_{i\notin S}(1-p_i))$

(b) Obvious: $\mathbb{P}(\text{System working}) = \mathbb{P}(\text{Bin}(n,p) \ge k)$

(c) $\mathbb{P}(1, n \text{ not working, System working}) = \mathbb{P}(\text{Bin}(n-2,p)\ge k)(1-p)^2$

Answer $= 1 - \dfrac{\mathbb{P}(\text{Bin}(n-2,p)\ge k)(1-p)^2}{\mathbb{P}(\text{Bin}(n,p)\ge k)}$

(4) (a) Obviously, $\mathbb{P}(|X| \le 0) = 0$. Let $x > 0$. Then

$\mathbb{P}(|X| \le x) = \mathbb{P}(-x \le X \le x) = \int_{-x}^{x} 1/_{2b}\, e^{-|t|/b}\,dt$

$\overset{\text{cov}}{=} 2\int_0^x 1/_{2b}\, e^{-t/b}\,dt = -[e^{-t/b}]_0^x = [e^{-t/b}]_x^0 = 1 - e^{-x/b}$.

So:

$F_{|X|}(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-x/b} & x > 0 \end{cases}$ & $f_{|X|}(x) = \begin{cases} 0 & x \le 0 \\ \frac{e^{-x/b}}{b} & x > 0 \end{cases}$

<u>Note:</u> $|X| \sim \text{Exp}(1/b)$.

(b) $\mathbb{E}[X^k] = \int_{-\infty}^{\infty} x^k e^{-x/b}/2b\, dx = 0$ if x is odd, as the integrand is an odd fn.

If k is even, $\mathbb{E}[X^k] = \mathbb{E}[|X|^k] = \int_0^\infty \frac{x^n}{b} e^{-x/b}\,dx$.

Let $I_k = \int_0^\infty \frac{x^n}{b} e^{-x/b}\,dx$; so $I_0 = 1$ and

$\int_0^\infty [x^n]\left[\frac{e^{-x/b}}{b}\right]dx \overset{\text{IBP}}{=} [-x^n e^{-x/b}]_0^\infty - \int_0^\infty -nx^{n-1}\frac{e^{-x/b}}{b}\,b\,dx = nb\int_0^\infty x^{n-1}e^{-x/b}\,dx$

$\Rightarrow I_n = (nb)I_{n-1} = n(n-1)b^2 I_{n-2} = \cdots = b^n n!\, I_0 = b^n n!$

$\Rightarrow \mathbb{E}[X^k] = \begin{cases} 0 & k \text{ odd} \\ k!\,b^k & k \text{ even} \end{cases}$

[Can use result on Gamma distribution's moments as well]

(5) $f_X(x) = \alpha x^{-\alpha-1}\mathbb{1}_{\{1<x\}}$.

(a) Clearly, $\mathbb{P}(X^2 \le 0) = 0$. Let $x > 0$.

$\mathbb{P}(X^2 \le x) = \mathbb{P}(X \le \sqrt{x}) = \int_1^{\sqrt{x}} \alpha t^{-\alpha-1}\,dt = [-t^{-\alpha}]_1^{\sqrt{x}} = 1 - x^{-\alpha/2}$

So,

$F_{X^2}(x) = \begin{cases} 0, & x \le 1 \\ 1 - x^{-\alpha/2}, & x > 1 \end{cases} \Rightarrow f_{X^2}(x) = \begin{cases} 0 & x \le 1 \\ +\frac{\alpha}{2}x^{-\alpha/2-1} & x > 1 \end{cases}$

(b) Clearly, $\mathbb{P}(X^{-1} \notin (0,1)) = 0$. So let $a \in (0,1)$.

$\mathbb{P}(X^{-1} \le a) = \mathbb{P}(X \ge a^{-1}) = \int_{a^{-1}}^{\infty} \alpha t^{-\alpha-1}\,dt = [-t^{-\alpha}]_{a^{-1}}^{\infty} = a^\alpha$

So,

$F_{X^{-1}}(x) = \begin{cases} 0 & x \le 0 \\ 1 & x \ge 1 \\ x^\alpha & x \in (0,1) \end{cases} \Rightarrow f_{X^{-1}}(x) = \begin{cases} 0 & x \notin (0,1) \\ \alpha x^{\alpha-1} & x \in (0,1). \end{cases}$

(c) Note that

$\mathbb{E}[(X^{-1})^k] = \alpha\int_0^1 x^{k+\alpha-1}\,dx = \alpha\left[\frac{x^{k+\alpha}}{k+\alpha}\right]_0^1 \qquad k+\alpha \ne 0$

$= \alpha[\ln x]_0^1 \qquad k+\alpha = 0$

Now it is clear that $\mathbb{E}[(X^{-1})^k] = \begin{cases} \alpha/k+\alpha & k+\alpha > 0 \\ \pm\infty & k+\alpha \le 0 \end{cases}$

So, $\mathbb{E}[(X^{-1})^k] < \infty \Leftrightarrow k+\alpha > 0$ and $= \alpha/_{k+\alpha}$ in that case.

Probability 2

A list of what you learn in prob 2, with a one-line "why".

- Review of prob 1.
- <u>Multi-variable</u> probability: To extend all prob. 1 notions to multi-variable situation.
- <u>Marginals</u>: Connecting m.v. probability with single variable probability.
- <u>Generating Functions</u>: Linking discrete combinatorial quantities with continuous quantities such as differentiable functions (generally, power series)
- <u>Conditional Prob.</u>: Extension to multi-variable situation.
- <u>Notions of Convergence</u>: Understanding various ways in which two random variables can be "close" to each other, or can be compared with eachother.
- <u>Central Limit Theorem and Laws of Large Numbers</u>: Understanding (only proving the weak LLN's finite variance case) these fundamental results in probability: that speak about the accuracy and error of estimation by repeated averaging.

<u>Good references</u>

→ Generating Functionology: Herbert Wilf

→ Kai Lai Chung's "Probability"

→ Klenke's Probability Theory: Read this to understand prob. 3 as well.

Good Luck!