11/11/21

Agenda: - Assignment 4 solutions + discussion.
- Combinatorial proofs.

(Q1) For the proof, we prove the inequality:
[Where $A = A_1 \cup \cdots \cup A_n$]

$$\mathbb{1}_A \le \sum_{i=1}^{n} \mathbb{1}_{A_i} - \sum_{i_1<i_2} \mathbb{1}_{A_i}\mathbb{1}_{A_j} + \cdots + \sum_{i_1<\cdots<i_{2k+1}} \mathbb{1}_{A_{i_1}} \cdots \mathbb{1}_{A_{i_{2k+1}}}$$

$$\mathbb{1}_A \ge \sum_{i=1}^{n} \mathbb{1}_{A_i} - \sum_{i_1<i_2} \mathbb{1}_{A_i}\mathbb{1}_{A_j} + \cdots - \sum_{i_1<\cdots<i_{2l}} \mathbb{1}_{A_{i_1}} \cdots \mathbb{1}_{A_{i_{2l}}}$$

Then, the inequality follows by summing both sides over $\omega \in \Omega$, and using the definition $P(E) = \sum_{\omega\in E} P(\omega) = \sum_{\omega\in\Omega} \mathbb{1}_E(\omega) P(\omega)$.

We prove this inequality by noticing that if $\omega \notin A$, then both sides of the inequality are zero. On the other hand, if $\omega \in A$, suppose that $\omega \in A_{i_1} \dots A_{i_k}$ for $i_1, \dots, i_k$ distinct. Then, we note that the number of times ω is counted on the RHS equals

$$k - \binom{k}{2} + \binom{k}{3} - \cdots \pm \binom{k}{d} = \sum_{i=1}^{d} (-1)^{i+1} \binom{k}{i}$$

[Where d is the level of the Bonferroni Inequality]
By induction, one can prove that

$$\sum_{r=0}^{l} (-1)^r \binom{p}{r} = (-1)^l \binom{p-1}{l} \quad \forall p, l.$$

Applying this, we can easily finish the problem. We take a small detour now.

__Combinatorial Proofs__

A combinatorial proof is an equation between two quantities, proved by counting the cardinality of some set in 2 different ways.

__Example__: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

This can be seen directly from the Binomial Theorem, but count the set $S = \{\text{Subsets of } \{1,2,\dots,n\}\}$.
- Each element can freely be chosen, or not chosen. That means 2 choices for each of n elements, so $|S| = 2^n$.
- Alternately, we can break the set S into subsets of size k: So that gives $\sum_{k=0}^{n} \binom{n}{k} = |S|$.

Thus, the identity is proved. (Next time, will include Pascal Δ).
It is possible to prove Q2,3,4 using a combinatorial proof.

(Q6) It is worth noting that the events B_m, C_m are very easy to describe, and it is quite clear that $\mathbb{P}(B_m)/\mathbb{P}(C_m) = 0/1$ (depending on m).
Using Q 2/3/4, it is easy to write down formulas for these quantities, and then prove that they equal 0 or 1 using binomial-theorem type arguments.

Some people reached the stage where they obtained an expression involving $\binom{\cdot}{\cdot}$, but were not able to simplify it. But arguments with the binomial thm. easily resolve this.

(Q7) This is done by noting that one can use Q4, with the logic of previous assignment's derangement qn.
However, we can also observe, that the event A_m is created by
- Fixing the indices where π, σ match.
- Deranging the rest of the indices.

Since the derangements can be found using the usual formula D_k, in this case, we can do this problem.
→ $|\Omega| = (n!)^2$. Fix σ. Then fix the m positions where σ, π match. Then derange the rest. So:

$$|A_m| = n! \binom{n}{m} D_{n-m}$$

(Q10)
(b) __Intuition: Even though__ A & B share the second coupon draw in common,
"The distribution of the second draw is independent of the occurrence of A".
So, A is going to be independent of B.
We can verify this by noting that we can restrict ourselves to the first three draws.
So $\Omega = \{(b_1, b_2, b_3) : 1 \le b_i \le n \ \forall i\}$.
$A = \{b_1 \ne b_2\} \Rightarrow |A| = n(n-1)n$.
For similar reasons, $B = \{b_2 \ne b_3\} \Rightarrow |B| = n(n-1)(n)$.

And now, comes the part where many, many people made a mistake.

The set $A \cap B$ is a set $A \cap B = \{b_1 \ne b_2, b_3 \ne b_2\}$.
"It is NOT the set of draws where ALL of b_1, b_2, b_3 are distinct, BECAUSE we only need $b_1 \ne b_2$ & $b_2 \ne b_3$: WE COULD HAVE $b_1 = b_3$".
So those, who wrote ~~$n(n-1)(n-2)$~~ [WRONG]
What is correct: b_2 can be chosen in n ways, and each of b_1, b_3 can be chosen in n-1 ways. (BECAUSE we __can__ have $b_1 = b_3$). So:
$|A \cap B| = n(n-1)(n-1)$.
Now, dividing by n^3 gives:

$$P(A) = \frac{n-1}{n}, \quad P(B) = \frac{n-1}{n}, \quad P(A\cap B) = \left(\frac{n-1}{n}\right)^2$$

In particular: $P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{\left(\frac{n-1}{n}\right)^2}{\left(\frac{n-1}{n}\right)}$
$= \frac{n-1}{n} = P(B)$. So B, __IS__ independent of A.

↦ In more detail, the common joint of A & B is the second draw.
- If A occurs, then the second draw is __still__ uniformly choosable from $\{1, \dots, n\}$.
- Even if A doesn't occur, then the second draw is unif. distributed from $\{1, 2, \dots, n\}$.

That, is why B is independent of A: all the information needed to assess B, have the same distribution regardless of whether A occurred or not. That is __independence__.

(c) This one is intuitively more difficult to recognize: so we do the math instead.
↦ __KEY POINT__: Work with the complements.
Why? Because A is the event that bin 1 contains a ball of label atmost 4, so one will have to break A into cases based on which ball is contained, and that leads to an inconvenient expression. Likewise B.
- With this knowledge, we can now find P(A). For this, look at A^c: A^c is the event that the 1^{st} bin contains balls only labelled 5 and above. But now, A^c can be counted by free choice; because balls 1,2,3,4 have 3 bin choices & 5,...,10 have 4 choices, so you get $|A^c| = 3^4 4^6$. Similarly, $|B^c| = 3^4 4^6$. What about $(A \cap B)$? Again, focus on $A^c \cap B^c$: this can be counted by free choice: balls 1,2,3,4 with 2 bin choices each & 5,...,10 with 6 choices each. So that gives $|A^c \cap B^c| = 2^4 4^6$. With the knowledge of $|\Omega| = 4^{10}$, we easily get:

$$P(A) = 1 - \frac{3^4 4^6}{4^{10}} = 1 - \left(\frac{3}{4}\right)^4 = P(B)$$

$$P(A\cap B) = 1 - P(A^c \cup B^c) = 1 - [P(A^c) + P(B^c) - P(A^c \cap B^c)] = 1 - 2\left(\frac{3}{4}\right)^4 + \left(\frac{2}{4}\right)^4$$

From here, $P(B|A) = \frac{P(A\cap B)}{P(A)}$ can be found.

__Key mistake__: Many people have computed P(A) directly, and there have been many misinterpretations.
- Some people equated A to the event that __exactly__ one ball with label less than 4 is in the first bin. That's a wrong interpretation. Some people equated A to the event that __all__ balls with label <4 lie in A. Some people interpreted the complement incorrectly: the complement is __NOT__ the event that A contains a ball of label 5 and above. All these are leading to the wrong answers.

__Intuition for non-independence__: You can see that A is not independent of B, though. What is common between A & B is the distribution of the balls labelled 1,2,3,4. IF A occurs, then it's clear that the distribution of the balls 1,2,3,4 is __SKEWED__ in favour of box 1, as opposed to A^c where it's skewed against box 1. So A & B, are not independent for sure.