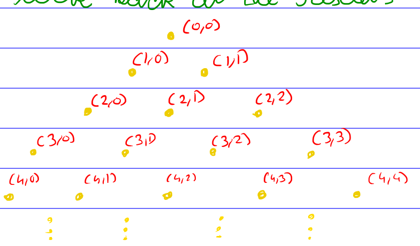
19/11/21

Agenda: – Some more combinatorics
– Assignment 5

(Keeping up with Joneses will be posted separately)

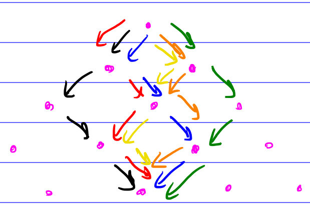
__Combinatorics__

We look back at the Pascal's triangle:

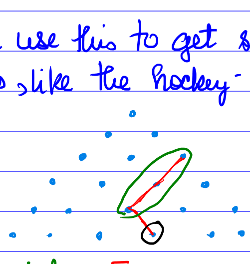


Remember the key point: the number of paths from (0,0) to (n,k) that go left-down or right-down at each point, equals $\binom{n}{k}$.
For example:

$\binom{4}{2}=6$
Highlighted in different colours



We can use this to get some very interesting identities, like the hockey-stick identity:



Sum of green = black

In binomials: $[n, r \in \mathbb{N}, n \geq r]$

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1} \; ; \; \sum_{j=0}^{n-r} \binom{j+r}{r} = \binom{n+1}{n-r}$$

__Proof:__ Any path to the end of the stick reaches exactly one of the points on the handle, after which it goes right. So the result is obvious from here.

__Also see__: The Fibonacci Connection:

$$F(n) = \sum_{k=0}^{n} \binom{n-k}{k}$$

Try to use the Pascal Triangle to prove this. Here's a hint: You only need to prove that if $G(n)$ is the RHS, then $G(n+1) + G(n) = G(n+2) \; \forall n$, & $G(0) = F(0)$, $G(1) = F(1)$. Use the triangle to see why these are true.

→ Also try: $\sum_{n=0}^{m} \binom{m}{n}^2 = \binom{2m}{m}$

[Hint: Think of some kind of "reflection" of the left/right-down path.]

$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$ (This one is more interesting with the pathwise perspective)

__Assignment 5 Questions__

(Q5) For (a) & (b), we have the following sample space:
$\Omega = \{(a,b) : a \in \{1,2,3,4,5\}, b \in \{H,T\}\}$
with 1,2 the double-heads, 3 the double-tail & 4,5 normal. Thus, we have:
$p(1,H) = 1/5$, $p(2,H) = 1/5$, $p(3,T) = 1/5$,
$p(4,H) = p(4,T) = 1/10$, $p(5,H) = p(5,T) = 1/10$
[and the rest have prob. 0]

(a) Note that we are "observing", in a sample space element, the __upper face__ of the coin in the first component of the tuple. So we have to isolate the lower face more carefully. 1,2 always have heads below, & 4,5 have heads below precisely when they are tails up.

$E_{HD} = \{(1,H),(2,H),(4,T),(5,T)\}$

$\Rightarrow \mathbb{P}(E_{HD}) = 1/5 + 1/5 + 1/10 + 1/10 = 3/5.$

(b) We must write down the "heads up" event for this.

$E_{HU} = \{(1,H),(2,H),(4,H),(5,H)\}$
$\Rightarrow \mathbb{P}(E_{HU}) = 3/5$
$\Rightarrow E_{HD} \cap E_{TD} = \{(1,H),(2,H)\}.$
$\Rightarrow \mathbb{P}(E_{HD} \cap E_{TD}) = 1/5 + 1/5 = 2/5$

$\Rightarrow \mathbb{P}(E_{TD} \mid E_{HD}) = \frac{2/5}{3/5} = 2/3.$

(c) We make sure the second coin toss is now included.
$\Omega^{(c)} = \{(a,b,c) : a \in \{1,2,3,4,5\}, b \in \{H,T\}, c \in \{H,T\}\}.$
We can write down the pmf now:
- $p(1,H,H) = 1/5$, $p(2,H,H) = 1/5$, $p(3,T,T) = 1/5$,
- $p(4,H,T) = p(4,T,H) = p(4,T,T) = p(4,H,H)$
$= p(5,H,T) = p(5,T,H) = p(5,T,T) = p(5,H,H) = 1/20.$

Let's write down the events of interest.
$E_{1HU} = \{(1,H,H),(2,H,H),(4,H,T),(4,H,H),(5,H,T),(5,H,H)\}$
[first heads up]
$E_{1HU} \cap E_{2HL} = \{(1,H,H),(2,H,H),(4,H,T),(5,H,T)\}$
[first heads up & second heads down]

We need $\mathbb{P}[E_{2HL} \mid E_{1HU}] = \mathbb{P}(E_{1HU} \cap E_{2HL}) / \mathbb{P}(E_{1HU})$
$= [1/5 + 1/5 + 1/20 + 1/20] / [1/5 + 1/5 + 1/20 + 1/20 + 1/20 + 1/20]$

$= 5/6$ ∎

Many people have made mistakes here. __All kinds of mistakes!__ List coming up later.

(Q10) If you're starting the MH problem from scratch, remember to ensure that all possible randomness is captured.
- The initial $P\alpha\alpha$ arrangement
- Which door you choose
- Which door Monty opens
- What the coin shows.

You have to be careful while writing down the sample space. The key point to keep in mind is this: Monty always picks a door with an α behind it, so if, by chance, you picked the correct door, then Monty has 2 choices of doors to open. The other key point, is __symmetry__: exploit the symmetry of the situation to write the space nicely.

$\Omega = \{(P\alpha\alpha,1,2,T),(P\alpha\alpha,1,3,T),(P\alpha\alpha,2,3,T),(P\alpha\alpha,3,2,T),$
$(P\alpha\alpha,1,2,H),(P\alpha\alpha,1,3,H),(P\alpha\alpha,2,3,H),(P\alpha\alpha,3,2,H)\}$
$\cup \{\text{"symmetric analogues"}\}$

What about pmf? For that, note that each of $P\alpha\alpha$, $\alpha P\alpha$, $\alpha\alpha P$ has a 1/3 weight, & your choice of door has a 1/3 on it as well. The H occurs with p chance, hence the tail with $1-p$ chance. As for Monty's choice: his choice is forced, if you took the wrong door: but split in half if you took the right door! With that:
$p(P\alpha\alpha,1,2,T) = 1/3 \times 1/3 \times 1/2 \times (1-p) = p(P\alpha\alpha,1,3,T)$
$p(P\alpha\alpha,1,2,H) = 1/3 \times 1/3 \times 1/2 \times p = p(P\alpha\alpha,1,3,H)$
$p(P\alpha\alpha,2,3,T) = 1/3 \times 1/3 \times (1-p) = p(P\alpha\alpha,3,2,T)$
$p(P\alpha\alpha,2,3,H) = 1/3 \times 1/3 \times p = p(P\alpha\alpha,3,2,H)$
(and for "symmetric analogues")

Now, you have won if you chose the correct door at the end. So that would be:
$E_{WIN} = \{(P\alpha\alpha,1,2,T),(P\alpha\alpha,1,3,T),(P\alpha\alpha,2,3,H),(P\alpha\alpha,3,2,H)\}$
$\cup \{\text{sym. analogues}\}$

$\Rightarrow \mathbb{P}(E_{WIN}) = 3 \times \left[\frac{(1-p)}{9} + \frac{2p}{9}\right] = \frac{1-p}{3} + \frac{2p}{3}$ ∎

If you forget to account for some sources of randomness, then you are getting the wrong answer here.

(Q4) Writing the sample space is easy: Pick a box, and pick a drawer. One can also pick a box and choose to segregate on the basis of what coins are in the drawers. So we have:
$\Omega = \{(b,d) : b \in \{1,2,3\}, d \in \{1,2\}\}$ OR
$\Omega' = \{(1,G),(2,S),(3,G),(3,S)\}$
In the latter case, the pmf $p(1,G) = p(2,S) = 1/3$ and $p(3,G) = p(3,S) = 1/6$.
Sticking with the former:

without loss of generality

$E_1 = \{(1,1),(1,2)\}$, $E_G = \{(1,1),(1,2),(3,1)\}$
$\Rightarrow \mathbb{P}(E_1 \mid E_G) = \mathbb{P}(E_1 \cap E_G)/\mathbb{P}(E_G) = 2/3$ ∎
This can be repeated with the other sample space as well.

(Q9) The simplest qn: the outcome is a bulb, and we record the manufacturer and the status of the bulb.
$\Omega = \{1,2,3\} \times \{G,B\}$ [Factory + Good/Bad]
$p(1,G) = 0.25 \times 0.95$; $p(1,B) = 0.25 \times 0.05$
$p(2,G) = 0.35 \times 0.96$; $p(2,B) = 0.35 \times 0.04$
$p(3,G) = 0.4 \times 0.98$; $p(3,B) = 0.4 \times 0.02$
Let $M_i = \{(i,G),(i,B)\}$, $i = 1,2,3$.
$E_B = \{(i,B) : i = 1,2,3\}$
$\Rightarrow \mathbb{P}(M_i \mid E_B) = \mathbb{P}(M_i \cap E_B)/\mathbb{P}(E_B).$
So we get:
$\mathbb{P}(E_B) = 0.125 + 0.140 + 0.080 = 0.345.$
and thus:
$\mathbb{P}(M_1 \mid E_B) = 0.125/0.345 = 25/69$
$\mathbb{P}(M_2 \mid E_B) = 0.140/0.345 = 28/69$
$\mathbb{P}(M_3 \mid E_B) = 0.080/0.345 = 16/69$
as desired.

We now discuss the mistakes that were made across this assignment from various people.

(Q9) A BIG ONE: The law of total probability states that for events $B_1, \ldots, B_n$ disjoint, with $\mathbb{P}(B_1 \cup \cdots \cup B_n) = 1$, and another event A,

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)$$

and __NOT__:

$$\cancel{\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \mid B_i)}$$

This is a big mistake. Otherwise, avoid silly & calculation mistakes.

(Q4) Make sure, if you're using Ω', then your prob. assignment is correct.

(Q10) There were almost no mistakes. If you wanted to simplify the analysis, then one can do this: the only outcomes to be recorded from the Monty Hall experiment is; whether, prior to tossing the coin, you were to win the game by switching, or win it by sticking.
So we can define:
$\Omega' = \{\text{you win by sticking, you win by switching}\} \times \{H,T\}$
$p(\text{win by sticking}, H) = p/3.$
$p(\text{win by switching}, H) = 2p/3.$
[Similarly replace p by $1-p$ for the other two]
So,
$\mathbb{P}(\text{winning}) = \mathbb{P}(\text{win by switching}, H) + \mathbb{P}(\text{win by sticking}, T)$
$= \frac{2}{3}p + \frac{1}{3}(1-p).$

(Q5) The bulk of the mistakes have come here. The major one being:
- If you pick a coin & toss it, and it lands up heads, then you know it can't be the tails-only coin. __The big mistake__ is to assume that this means the prob. of the other 4 coins are 1/4 each. __WRONG__
Why? Because 2 coins are 2-headed: so in total, there are 6 heads across the coins. So, given that you saw a head, that head comes from the double-headed coins with prob. 2/6 each, and from the normal coins with prob. 1/6 each.

$\cancel{\{1/4, 1/4, 1/4, 1/4\}}$ $\{1/3, 1/3, 1/6, 1/6\}$ ✓

This uniformity was the biggest error. Taking uniform distribution leads __wrongly__ to 3/4.

Apart from this, I only saw silly & sample space errors. Please be careful when you are assigning pmfs.