29/10/21

Agenda for today:

- Assignment 3 Answers (2,4,6,8).
- Assignment 4 clarification
- Additional Problems / Ross.

(Q6) $C = \{1, 2, \dots, 52\}$

$\sigma : C \to C$ bijection is a shuffling of a deck.
So if $S_{52} = \{\sigma : C \to C \text{ bijection}\}$ then $\Omega = S_{52} \times S_{52}$. $|\Omega| = (52!)^2$
$E_1 = \{\text{first cards match}\} = \{(\sigma, \eta) : \sigma(1) = \eta(1)\}$
$E_2 = \{\text{second cards match}\} = \{(\sigma, \eta) : \sigma(2) = \eta(2)\}$
To find $P(E_1 \cup E_2)$, use PIE:

$$P(E_1 \cup E_2) = \underbrace{P(E_1) + P(E_2) - P(E_1 \cap E_2)}_{\text{can be calculated using similar ideas}}$$

Let's look at E_1:

$$E_1 = \{(\sigma, \eta) : \sigma(1) = \eta(1)\}.$$

Then E_1 is chosen by choosing bijections σ', η' on $C \setminus \{\sigma(1)\}$, and extending them to σ, η on C. The choice of σ', η' is available in $(51!)^2$ ways, and $\sigma(1) = \eta(1)$ can be chosen in 52 ways, so $(51! \times 52!) = |E_1|$. So $|E_1| = 1/52$.
The rest $E_1, E_1 \cap E_2$ are similar.
Qn: Can we say "If we are choosing (σ, η) uniformly from Ω, then we are choosing uniformly $(\sigma(1), \eta(1))$ from $C \times C$, so the answer is $52/52 \times 52 = 1/52$?"
Ans: As long as you justify the quoted statement, this logic is also ok.

(Q2) Ω is definable as in class:

$$\Omega = \{(n_1, \dots n_r) : n_i \in \{1, 2, \dots, n\}\}, \ |\Omega| = n^r$$

$\forall \omega \in \Omega$, $p(\omega) = 1/|\Omega| = 1/n^r$.
For the event in question, if $r_1 + \dots + r_n \neq r$ then $P(E) = 0$.
Else, E is an event representing "permutations with repeated letters". This is solved in two ways:

(i) Pick r_1 spots for 1s. Then pick r_2 out of $r - r_1$ spots for 2s, then r_3 spots out of $r - r_1 - r_2$ spots for 3s, etc, so:

$$\binom{r}{r_1}\binom{r - r_1}{r_2}\binom{r - r_1 - r_2}{r_3} \cdots \binom{r_n}{r_n} = \frac{r!}{r_1! r_2! \cdots r_n!}$$

(ii) Imagine that the 1s, 2s etc. are distinct: so we permute and then identify them. So we get $r!$ permutations, and then account for the identification of 1s, 2s as separate, by dividing by $r_1!, r_2!$ etc., giving the same answer.

(Q4) A similar qn. to 6, you can take $E_1 = \{(0, r_2, \dots, r_n)\}$, $E_2 = \{(r_1, 0, \dots, r_n)\}$, so $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ and definition of $P(E) = \sum_{x \in E} p(x)$, we can easily finish. Also note that $\sum_{x \in \Omega} p(x) = 1$ is a direct consequence of the multinomial theorem.

(Q8) $\Omega = \{\text{permutations of } \{1, 2, \dots, n\}\}$
$E = \{\sigma(i) \neq i \ \forall i = 1, 2, \dots, n\}$
We have 2 approaches.

(i) Take $E^c = \{\exists i : \sigma(i) = i\} = \bigcup_{i=1}^{n} A_i$ where $A_i = \{\sigma(i) = i\}$.
We can find $P(E^c)$ using P.I.E., indeed note that
$1 \leq i_1 < \dots < i_k \leq n \Rightarrow A_{i_1} \cap \dots \cap A_{i_k} = \{\sigma(i_1) = i_1 \dots \sigma(i_k) = i_k\}$
But, like in Q_2, a perm. in ↓ is a permutation on the set $\{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$, suitably extended. So, we get that $P(A_{i_1} \cap \dots \cap A_{i_k}) = (n-k)!/n!$. Applying the PIE, we get:

$$P(E^c) = \underbrace{[P(A_1) + \dots + P(A_n)]}_{1/1!} - \underbrace{[P(A_1 \cap A_2) + \dots + P(A_{n-1} \cap A_n)]}_{1/2!} + \underbrace{[P(A_1 \cap A_2 \cap A_3) + \dots + P(A_{n-2} \cap A_{n-1} \cap A_n)]}_{1/3!}$$

$$= 1 - 1/2! + 1/3! \dots \pm 1/n!$$

Hence,

$$P(E) = 1/2! - 1/3! + \dots \pm 1/n!$$

(ii) We create a recursion: consider sample spaces corresponding the problem with n ladies, so you have (Ω_n, p_n) with events E_n. We create a recursion on $P(E_n)$ ← To be understood as the pmf on Ω_n.
Recognize $P(E_n) = |A_n|/n!$ where A_n = no. of derangements of n objects. To calculate A_n:
For $\sigma \in E_n$, $\sigma(1)$ must equal one of $2, \dots, n$ with equal likelihood, so let $\sigma(1) = j$. Then:
- Either $\sigma(j) = 1$. If so, then ensure that $\{1 \dots n\} \setminus \{j\}$ is deranged – A_{n-2}.
- Or $\sigma(j) \neq 1$, then we actually return to case $n-1$, because the situation is the same: we want $\sigma(2) \neq 2, \sigma(3) \neq 3 \dots \sigma(n) \neq n$, so we get A_{n-1} from here.

So: we land up with $A_{n-2} + A_{n-1}$ outcomes for each of $n-1$ choices i.e.

$$A_n = (n-1)((A_{n-1}) + (A_{n-2}))$$

Dividing by $n!$ to change to E_n,

$$P(E_n) = P(E_{n-1}) + (n-2) P(E_{n-2}).$$

Using $P(E_1), P(E_2)$, one can derive the recursion by induction.

Common Mistakes

(Q6) – It is true that $E_1, E_2, E_1 \cap E_2$ are events whose probabilities can be calculated using similar logics. However, it is not true that E_1, E_2 are independent.
To give a brief reason why, if the first cards do match, then the second cards' probability of matching becomes 1/51, different from the 1/52 it was in isolation.
because $\sigma(2) \in \{2, 3, \dots, 52\}$ unif. as the choice of $\sigma(2) = 1$ is gone.
This error was rare.
– For $E_1 \cap E_2$, make sure you realize when choosing cards for $\sigma(1), \sigma(2)$, that order matters:
– If you wrote $P(E_1) = 1/52 \cdot 52$ and likewise, you forgot to account for the choice of $\sigma(1)$.
– Some people forgot there were 2 piles!

(Q2) – The biggest mistake some people made was to assume the model was with indistinguishable balls. It had distinguishable balls, so it matters, which balls go in which bin.
– The question was, for a fixed vector $(r_1, \dots r_n)$, to find the prob. of the event that r_i balls are in box i, $i = 1, \dots, n$. Some people ended up misreading this & varying the r_i. The answer needs to depend upon the r_i, right?

(Q4) – Always remember: if E is an event, of a prob. space (Ω, p), then:

$$P(E) = \sum_{x \in E} p(x)$$

In doing this problem, you must:
(i) make sure that the E you write down, is actually a subset of Ω as described.
(ii) Use the formula to find $P(E)$.

So you should write e.g.:

$$E_1 = \{(0, r_2, \dots, r_n) : r_2 + \dots + r_n = r\}$$

$$P(E_1) = \sum_{r_2 + \dots + r_n = r} p((0, r_2, \dots, r_n))$$

and proceed. (i) & (ii) are the major talking points and where people made mistakes.

(Q8) NO ONE made a mistake here, that is good to know!