# Assignment 1 Solutions

October 13, 2021

## Question 1

The answers are as follows :

- One coin toss : Two outcomes.
- Rolling a dice : Six outcomes.
- Toss coin $n$ times : $2^n$ outcomes.
- The experiment could be rolling an $n+1$-sided dice, and the cardinality of the set is $n+1$.
- Selecting a card from a pack : 52 outcomes
- Poisitioning labelled particles : $n^r$
- Configuration of unlabelled particles : using a stars-and-bars argument (check MSE for what this is) we can show that this equals $\binom{n+r-1}{r}$.
- Unlabelled particles with no two particles at the same site : you just choose $r$ sites out of $n$, so $\binom{n}{r}$ is the answer.

## Question 2

In each example, let $E_i$ denote the event that the $i$th cell/site is empty. Then, no two consecutive sites being occupied is equivalent to every pair of consecutive sites having at least one empty site. Hence for each $1 \leq i \leq n-1$, either $E_i$ occurs or $E_{i+1}$ occurs i.e. our desired event is $\cap_{i=1}^{n-1}(E_i \cup E_{i+1})$. It is easy to write down what $E_i$ is for each example, and we are done.

## Question 3

The sample space is just $\Omega = \{0,1\}^n$, the set of all $n$ bit strings in 0 and 1. Let, for a string $S$ comprised of letters in $\{0,1\}$ , $\Sigma_S$ be the sum of all the letters in $S$. Note that for $S \in \Omega$ the length of $S$ is $n$,therefore $\Sigma_S < \frac{n}{2}$ indicates that there are more zeros than ones in the message. Using this, we can write the other event.

## Question 4

The answer can be given as follows. For $u \in \{0,1\}$ let $u_k$ be the $k$ bit string given by $u_k = \overline{uu\ldots uu}$ with $u$ repeated $k$ times. Then :

$$\Omega = \{(a^1)_k(a^2)_k \ldots (a^n)_k : a^i \in \{0,1\} \text{ for all } 1 \leq i \leq n\}$$

can be a sample space.

## Question 5

### a

First, mathematically represent a coupon by the notation $a_k$ where $a \in \{1,...,10\}$ and $k \in \{1,...,a\}$. Using this, you can easily write the sample space. Since $a$ is the label of $a_k$, both events can now easily be written down.

### b

Let $E$ be a set of five elements, meant to be people. Then, a pair in $E$ is denoted by $(a,b)$ for $a \neq b \in E$. Let $S$ denote the set of all pairs in $E$. The idea of the experiment, is to analyse the existence of friendship/non-friendship (won't go so far as to call it enemity!) between *every* pair of friends. What this means, is that EVERY *outcome* of the experiment being performed on the set of friends, is the complete set of information about whether each pair is friends or not.

So for example, every pair could be friends. Or no pair could be friends. Or exactly one pair could be friends and the rest could be non-friends. How does one mathematically describe an outcome? It's simple : collect all the pairs of friends together, and present it as an outcome. In other words, every subset of $S$ actually describes an outcome, because if $T$ is a subset of $S$, then every pair of $T$ being friends and every pair not in $T$ being non-friends represents a reasonable outcome of the experiment. Therefore, the sample space is $\Omega = \{T : T \text{ is a subset of } S\}$.

What about the event of three people being friends with each other? Fix three distinct people $a,b,c \in E$. Consider the event $E_{a,b,c}$ as the subset of $\Omega$ where $a,b,c$ are friends with each other. The event we want is then $E = \cup_{a\neq b\neq c \in E} E_{a,b,c}$ , as can be clearly seen.

What is $E_{a,b,c}$? We want $a,b,c$ to be friends with each other, so the pairs $(a,b),(b,c),(c,a)$ must be friends. So any favourable outcome must contain these three pairs. In other words, $E_{a,b,c} = \{T \subset S : (a,b) \in T, (b,c) \in T, (c,a) \in T\}$ and we can finish.

### c

Let $(a,b)$ for $a,b \in \{1,...,8\}$ be used to denote a square of the chessboard. Naturally an outcome is a *pair* of positions i.e. a subset of $\{\{(a,b),(c,d)\}$ :

$a, b, c, d \in \{1, ..., 8\}\}$, so you need to find for what $(a, b)$ and $(c, d)$ is it true that kings placed on $(a, b)$ and $(c, d)$ don't attack each other. Naturally from our knowledge of king moves, this occurs if and only if $(c, d)$ isn't horizontally ,vertically or diagonally adjacent (or equal!) to $(a, b)$, so you can easily write the said event down.

### d

You can have something like $\Omega = \{(x, y, z) : x \in \{A, B\}, y \in \{B, C\}, z \in \{C, A\}\}$. But try this for a matrix representation : create a $3 \times 3$ matrix with rows labelled $A, B, C$. Then, in the entries $i, j \in \{A, B, C\}$, if $i = j$ then insert 0 (or anything) and for $i \neq j$ insert the match winner between $i$ and $j$ (which will be in $\{A, B, C\}$). This leads to a symmetric matrix which encodes an outcome of all three games.

### e

This is generalization of the previous question. A clear winner can be described by looking at every row in the matrix. Obviously, 'no clear winner' is the opposite of 'there is a clear winner' so both events can be written down.

## 6

Let $W_k = R_k^c \cap G_k^c$. Then

### a

This is just $W_1$.

### b

This is $G_1 \cap (G_2 \cup R_2)$.

### c

Break the event on the basis of what color the first and last ball end up being : for example, for red the event is $R_1 \cap R_n$. Now change the color.

### d

Break the event on the basis of which color comes in which draw. For example, green first, red second and white third is $G_1 \cap R_2 \cap W_3$. Now change the order in which the colors appear to all possible orderings of $\{1, 2, 3\}$.