QUICKSORT ALGORITHM:

Suppose $a_1, a_2, \ldots a_n$ are n numbers and you want to arrange them in incr. order.

Algorithm?

If $a_2 < a_1$ then arrange as $a_2 a_1 a_3 \cdots a_n$.

Compare a_3 with a_2 & a_1 and place it suitably.

After stage k, $a_1, \ldots, a_k$ are arranged in correct order.

Goal: Minimize # of comparisons.

Ex** calculate best and worst.

A different algorithm.

Pick a_k – Let S_1 be the set of a_i's smaller than a_k

Called pivot.

Let S_2 . . . larger than a_k.

Arrange as $S_1 \; a_k \; S_2$ [S_1, S_2 are ordered as in original].

– Now apply the procedure above to S_1 & S_2.
until S_1 & S_2 are singletons.

Eg. $n \;\; n-1 \;\; n-2 \;\; \ldots \;\; 1$.

Leftmost selection:

$n-1 \;\; n-2 \;\ldots\; 1 \;\; n$ (n-1 comparisons)

↓

$n-2 \;\ldots\; 1 \;\; n-1 \;\; n$ (n-2 comparisons)

↓

⋮

$1 \;\; 2 \;\ldots\; n-2 \;\; n-1 \;\; n$ (1)

$$\# \text{ comparisons} = (n-1) + (n-2) + \ldots + 1 = \frac{(n-1)n}{2}$$

Rightmost selection is similar.

Middle selection (n-1 comparisons)

$\frac{n}{2}-1, \ldots 1, \frac{n}{2}, n, n-1, \ldots, \frac{n}{2}+1$

$\downarrow$

$\vdots$ (n-2 comparisons)

Ex** Show that $2\log_2 n$ steps suffice.

But we are in a prob. course & so make randomized selection of pivots.

$X(\omega) = \#$ comparisons

But what is ω? What prob. space?

Given $a_1, \ldots, a_n$ let $y_1, \ldots, y_n$ be the arrangment in incr. order.

$\Omega' = \{$ possible set of all pairs (y_i, y_j) compared by the algorithm $\}$

Let y_{k_0} be first pivot

$\{(y_1, y_{k_0}), \ldots (y_n, y_{k_0})\} - (y_{k_0}, y_{k_0}) \subseteq \omega' \in \Omega'$

Let y_{k_1}, y_{k_2} be the pivots in Round 2.
with $k_1 < k_0 < k_2$

$\{(y_{k_1}, y_1), \dots, (y_{k_1}, y_{k_0-1})\} - (y_{k_1}, y_{k_1}) \subseteq W'$

$\{(y_{k_2}, y_{k_0+1}), \dots, (y_{k_2}, y_n)\} - (y_{k_2}, y_{k_2}) \subseteq W'$.

&

so forth.

Another simpler representation.

$W = (y_{k_0}, y_{k_1}, y_{k_2}, \dots, y_{k_{n-1}})$ the sequence of pivots.

This suffices to reconstruct the algorithm. (why?)

X = # comparisons made.

$X_{i,j} = \mathbb{1}[y_i$ & y_j were compared by the algorithm$]$

$$X = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{i,j}$$

$$\mathbb{E}X = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbb{E}X_{i,j}$$

$\mathbb{E}X_{i,j} = \mathbb{P}(y_i$ & y_j were compared by the algorithm$)$

If y_i & y_j were compared then one should've been chosen as the pivot & the other was in the same list.

$$y_1 \cdots y_{i-1}, y_i, y_{i+1}, \cdots, y_j, y_{j+1} \cdots y_n$$

This means all the pivots before y_i or y_j should not be in $\{y_{i+1}, \cdots, y_{j-1}\}$.

Or equivalently,

the first pivot from $\{y_i, \cdots, y_j\}$ should be y_i or y_j

$$\mathbb{P}\{\text{first pivot from } \{y_i, \cdots, y_j\} \text{ is } y_i \text{ or } y_j\}$$

$$= \frac{2}{j-i+1}$$

So $$\mathbb{E}X = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = 2 \sum_{i=1}^{n} \sum_{k=2}^{n-i+1} \frac{1}{k}$$

$$\leq 3 \sum_{i=1}^{n} \log(n-i+1) \leq 3 n \log n$$