Indian Statistical Institute

Analysis II (HW - 1) Date: February 7, 2022 Instructor : Jaydeb Sarkar

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NOTE: (i) $B[a,b]$ = the set of all bounded real-valued functions on $[a,b]$. (ii) $R[a,b]$ = the set of all Riemann integrable functions on $[a,b]$. (iii) $C[a,b]$ = the set of all continuous functions on $[a,b]$. (iv) $C[a,b]$ = the set of all partitions on $[a,b]$.

(1) Let $f \in B[a,b]$. Prove that f is a constant function if and only if there exists $P \in P[a,b]$ such that $L(f,P) = U(f,P)$.

(2) Give an example of a function $f \in B[0,1]$ such that $f \notin R[0,1]$ but $f^2 \in R[0,1]$.

(3) Let $f, g \in B[a,b]$, and let $f(x) \leq g(x)$ for all $x \in [a,b]$. Prove that

$$\underline{\int_a^b} f \leq \underline{\int_a^b} g \quad \text{and} \quad \overline{\int_a^b} f \leq \overline{\int_a^b} g.$$

(4) True/False (with explanation)? "If $f(x) \leq g(x) \leq h(x)$ for all $x \in [a,b]$, and $f, h \in R[a,b]$, then $g \in R[a,b]$."

(5) Consider the characteristic function $\chi_{[1,3]}$ on $[0,5]$ (that is, $\chi_{[1,3]} : [0,5] \to \mathbb{R}$ where $\chi_{[1,3]}(x) = 1$ if $1 \leq x \leq 3$ and $\chi_{[1,3]}(x) = 0$ if $3 < x \leq 5$). Prove that $f \in R[0,5]$ and compute $\int_0^5 \chi_{[1,3]}$.

(6) Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \text{ and } \frac{1}{x} \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is not continuous at n, $n \in \mathbb{N}$. Is $f \in R[0,1]$?

(7) Let $f \in R[a,b]$ be a nonnegative function. If $f(r) = 0$ for all $r \in [a,b] \cap \mathbb{Q}$, then prove that $\int_a^b f = 0$.

(8) (i) Give an example of two functions $f, g \in B[a,b]$ such that $f, g \notin R[0,1]$, but $fg \in R[a,b]$. (ii) Give an example of two functions $f, g \in B[a,b]$ such that $f \in R[a,b]$, $g \notin R[a,b]$, but $fg \in R[a,b]$.

(9) (a) (Cauchy-Schwarz inequality) Let $f, g \in R[a,b]$. Prove that

$$\Big(\int_a^b fg\Big)^2 \leq \Big(\int_a^b f^2\Big)\Big(\int_a^b g^2\Big).$$

[Hint: Expand $\int_a^b (tf+g)^2$ into a quadratic in t. Clearly $\int_a^b (tf+g)^2 \geq 0$ for all $t \in \mathbb{R}$. Now look at the non-positive discriminant.]

(b) Given that f and g are continuous, prove that equality holds if and only if one of the functions is a constant times the other.

(c) Use the Cauchy-Schwarz inequality for an upper estimate on the integral

$$\int_0^{\frac{\pi}{2}} \sqrt{x \sin x}\, dx.$$

[By the way, can you compute the value of the integration?]

(10) (Minkowsky's inequality) Let $f, g \in R[a,b]$. Prove that

$$\Big(\int_a^b (f+g)^2\Big)^{\frac{1}{2}} \leq \Big(\int_a^b f^2\Big)^{\frac{1}{2}} + \Big(\int_a^b g^2\Big)^{\frac{1}{2}}.$$

[Hint: Note that $\int (f+g)^2 = \int f^2 + \int g^2 + 2\int fg$. Now consider the Cauchy-Schwarz inequality.]

(11) Let $f \in R[0,1]$. Prove that

$$\int_0^1 f = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(\frac{j}{n}).$$

(12) Use (11) to find the limit

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{m=1}^{n} \frac{1}{\sqrt{m}}.$$

(13) Let $f \in B[a,b]$ and let $P \in P[a,b]$. Prove that (i) $U(f,P)$ is the supremum of the set of all Riemann sums of f over P, and (ii) $L(f,P)$ is the infimum of the set of all Riemann sums of f over P.

(14) Let $f \in C[a,b]$ and let $P \in P[a,b]$. Prove that $U(f,P)$ and $L(f,P)$ are Riemann sums of f over P.