Indian Statistical Institute

Analysis II (HW - 10) Date: April 19, 2022 Instructor : Jaydeb Sarkar

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(1) Discuss the convergence behavior (double and iterated) of the sequence $\{a_{m,n}\}_{m,n\geq 1}$, where $a_{m,n} =$

$$(i)\ \sin\frac{m}{n},\ (ii)\ \frac{mn}{(m+n)^2},\ (iii)\ \frac{mn}{m^2+n^2},\ (iv)\ m^{-\frac{1}{n}}.$$

(2) Let $\alpha_n \to \alpha$ and $\beta_n \to \beta$, and suppose $a_{m,n} = \alpha_m \beta_n$ for all m and n. Prove that $\{a_{m,n}\}_{m,n}$ is a convergent double sequence, and

$$\lim_{m,n} a_{m,n} = \lim_m(\lim_n a_{m,n}) = \lim_n(\lim_m a_{m,n}) = \alpha\beta.$$

(3) State and prove a comparison test for double series with nonnegative terms.

(4) Suppose $a_{m,n} = (-1)^m n^{-m-2}$ for all $m \geq 0$ and $n \geq 2$. Prove that

$$\sum_{m\geq 0, n\geq 2} |a_{m,n}| = 1 \text{ and } \sum_{m\geq 0, n\geq 2} a_{m,n} = \frac{1}{2}.$$

(5) Let $\sum_{m,n} a_{m,n}$ be absolutely convergent, and let $\eta : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ be a one-to-one mapping. Prove that

$$\sum_n a_{\eta(n)},$$

converges.

(6) Let $\sum_{m,n} a_{m,n}$ be absolutely convergent, and let $\eta : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}$ be a bijection. Prove that the reordering of the double series

$$\sum_{m,n} a_{\eta(m,n)},$$

is also absolutely convergent. Moreover, prove that

$$\sum_{m,n} a_{m,n} = \sum_{m,n} a_{\eta(m,n)}.$$

(7) Discuss the convergency of the double series

$$\sum_{m,n\geq 1} \frac{1}{mn^4}.$$

(8) Discuss the convergency of the double series

$$\sum_{m,n\geq 1} \frac{1}{(m+n)^p},$$

converges if and only if $p > 2$.

(9) Let $\sum_n \alpha_n$ and $\sum_n \beta_n$ be two convergent series. Prove that

$$\sum_{m,n} \alpha_m \beta_n = (\sum_m \alpha_m)(\sum_n \beta_n).$$

(10) Let $r \in \mathbb{R}$. Prove that $\sum_{m,n=1}^{\infty} r^{mn}$ converges if and only if $|r| < 1$. In this case, prove that the iterated series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} r^{mn}$ converges. Also, prove that

$$\sum_{m,n=1}^{\infty} r^{mn} = \sum_{n=1}^{\infty} \frac{r^n}{1-r^n} \qquad (|r| < 1).$$